ME525 Applied Acoustics Lecture 1, Winter 2022

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About class notes

- Use class website to download lecture notes, supplementary material, homework and upload assignments
- Notes posted for each lecture either before, or within a couple of hours after the lecture
- Typos happen (if you see them tell me!) Corrections posted asap
- There is no textbook. Many references to texts provided in the notes
- I can recommend several texts depending on your interests

Introduction

ME 525, Applied Acoustics, is intended to introduce acoustics through its various applications and sub-fields, such as medical ultrasound, sonar and underwater sound, noise studies, and to introduce new perspectives on data analysis that can be applied to your particular field of research. We emphasize *linear* acoustics, an effect that encompasses the majority of our everyday experience with sound along with many applications of acoustics.[1]

Linear Acoustics I: signals add without distortion

So what is meant by linear ? Let's evaluate from the perspective of our everyday sensation of sound. I made this recording (Fig. 1) on my iphone on awalk near Seattle's Seward Park. The sound is of my own foot steps, plus a bit of laughing and talking by others walking nearby, plus a ar stereo "booming out" a very loud low-frequency sound. One tends to "feel" this sound as a vibration rather than hear it, as the frequency is getting low enough such that humans (or at least old guys like me) have a hard time actually hearing this sound. The middle figure shows a expanded scale of covering 0.5 sec– count the number cycles, ~ 20, so the frequency is about 40 Hz.

I can easily play this sound but you'll not hear the low-frequency car stereo as its difficult for PC speakers broadcast lower frequencies with reasonable fidelity. But you would certainly hear the people talking. (For those interested a wav file, representing a 15-s sample of this signal is available in the supplementary folder for this lecture. To examine the data in matlab use: `[signal,freq] = audioread('LFsig.wav');`. The data is contained in `signal` and the sampling frequency identified by `freq`. Again, only if you your're interested now–we may come back to this data later.)

[1]Numerous applications in acoustics involve non-linear acoustic phenomena such, shock waves and sonic booms, and some applications in medical ultrasound. We won't have time in this course to enter into this line inquiry, but in any case, a solid foundation in linear acoustics is essential to proceed.

The bottom plot is shows result of high-pass filtering to remove this low-frequency signal. Were you to liston again the filtered signal representing the talking sounds identical to the original, non-filtered, recording. The two signals: (1) the annoying low-frequency car sound of high volume, and (2) the sounds of people walking, add together without distortion. This is the result linearity where *linear superposition* of the two signals has occurred. Such linear superposition is also easily observed on the water for linear gravity waves when two boats pass, their separate boat wakes interact briefly which can produce a complex pattern, after which the wave field from each boat continues undisturbed.

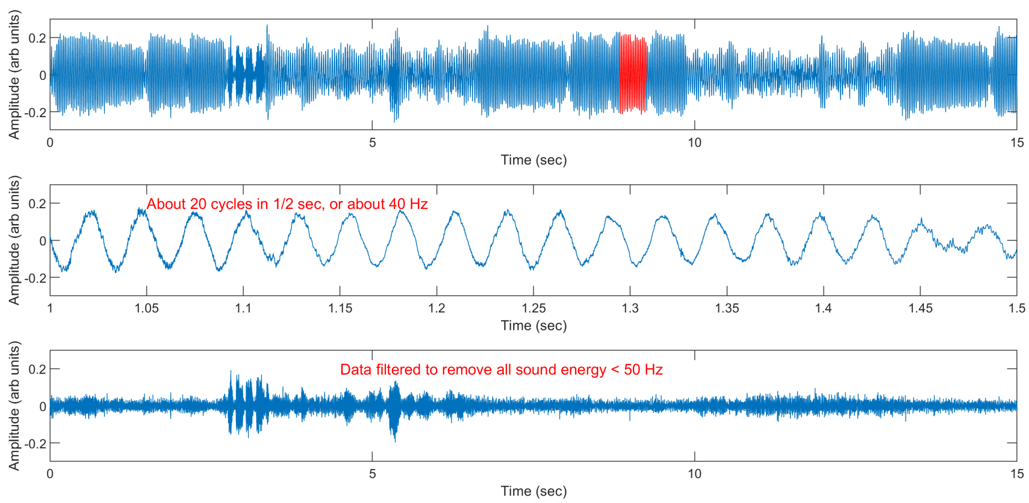

Figure 1: Top: Original signal recorded on my iphone. Red region highlights a very loud segment "narrow band" noise coming from a car stereo about 200 m away. Middle: expanded segment illustrating that frequency of the loud segments is about 40 Hz. Bottom: data filtered to remove all frequencies below 50 Hz.

Another view is of these data is the frequency spectrum (Fig. 2). Acoustic data as you can now well imagine with case of the care stereo is often in the form of distribution of acoustic frequencies, characterized in the form a spectrum or more formally, the spectral density. We'll have occasion to work this issue later in course, but for now notice the upper plot showing a spectrum that appears to be dominated by one narrow band of frequencies, which originates from the car. This is indeed true, but the linear plot cannot give the full picture, and very often in acoustics a log-scale (bottom plot) is needed to show the full picture. Notice the car stereo and the talking sounds are separated by nearly four-orders of magnitude! Only a log-scale can properly convey such information– keep this mind as we move forward in this class. Use of a log-scale to convey the enormous *dynamic range* often observed in the acoustic data is the same as expressing such data in *decibels*, abbreviated as dB. We'll also have more to say about dB.

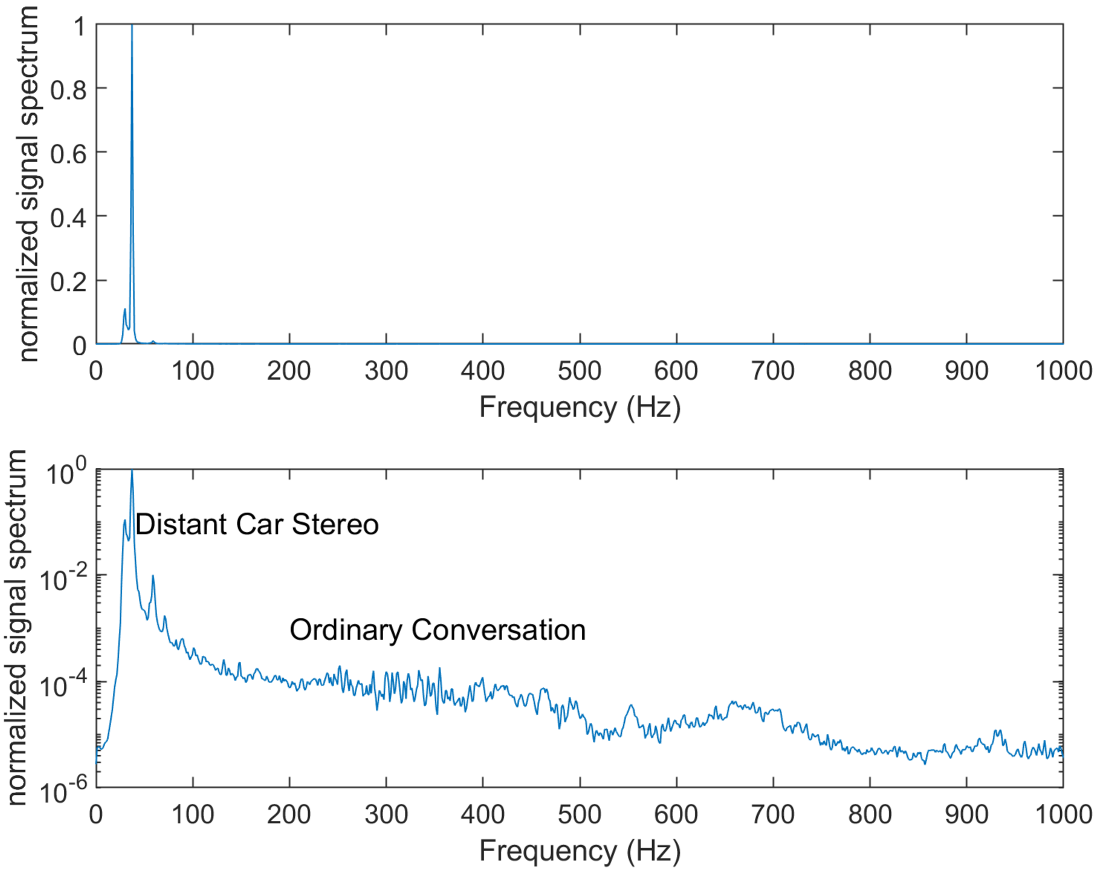

Figure 2: Top: Normalized variance spectrum of the recording plotted in linear scale Bottom: same data but now plot in log scale for y-axis that shows the more clearly variance contributions from car stereo and ordinary conversation.

Linear Acoustics II: acoustic variables are small compared to their ”background” state

Acoustic pressure and acoustic velocity

Sound is a mechanical wave, which might also be called a pressure wave, because sound pressure is transmitted via wave action through the medium such as air or water. Such a wave shares many attributes with waves on the surface of water, but instead of alternate crests (high water) and troughs (low water), with sound there are alternate regions of compression (high pressure) and expansion (low pressure) of the medium. For water waves the restoring force is gravity, for sound waves the restoring force is the fluid’s compressibility, or the opposition that the fluid exhibits against being compressed. Once the fluid is compressed it will return to its original state. But inertia of the fluid mass causes the fluid motion to ”overshoot” this state and reach an expanded state. The fluid’s compressibility property again acts on this but in reverse and the process of alternate regions of compression and expansion (also known as rarefaction) continues. In summary the fluid mass (density) and spring constant (compressibility) are needed to produce wave motion.

In this course we'll be discussing *acoustic variables* as applied to fluid media such as air (or gases), water, and to a good approximation, many biological tissues. Sound waves in fluid medium are composed of *compressional waves*, representing the alternative high/low state of compression. Similar compressional waves apply to a solid media, but in addition, solid media can have shear waves. For example, seisometers would be used to measure both compressional and shear wave effects in the solid earth.

Pressure is by far the most important acoustic variable describing sound, e.g., as detected by our ear drums, microphones, hydrophones, etc. Pressure is the *dynamic* part of the *sound field*. But there are also *kinematic* parts: acceleration, velocity, displacement, all involving motion of the medium and intimately linked with pressure.

To describe the variation of the dynamic and kinematic quantities in space and time associated with a sound wave, we need an acoustic wave equation. We'll proceed to derive this by relying on the assumption of linearity which in turn depends on the smallness of these quantities as related to their background state. Many solutions of the acoustic wave equation will already be familiar, such simple harmonic motion.

Call the pressure in the absence of sound a steady state, or background, pressure, p_0. For example, underwater $p_0 = P_{atm} + P_{gauge}$ representing all the pressure of the atmosphere (about 10^5 Pa as sea level) plus the constant increase in pressure with depth (the gauge pressure) which equals $\rho_0 g H$ where H is depth and g is gravitational constant (keep everything in MKS!) In air the background pressure is $p_0 = P_{atm}$.

Identify p_1, as the acoustic pressure caused by a sound wave passing through a region of the medium (water or air) and it represents a very small "perturbation" about p_0 such that

$$p_1/p_0 << 1 \tag{1}$$

We show later that the length scale of that region over which the perturbation exists, equals sound *wavelength*.

Let's test: my voice if reaching you in an ordinary classroom setting would be about 0.02 Pa so with $p_o = 10^5$ Pa, we for sure have $|p_1|/|p_0| << 1$. Incidentally this 0.02 Pa represents about 60 dB–but you will have to get used to the ways and means of decibel convention (which is different for air and water). The *reference pressure* for air is 20 μ Pa. See if you can figure out how 0.02 Pa translates to 60 dB. Similarly, the "threshold of pain" in for human hearing is about 2 Pa (or 100 dB with reference to 20 μ Pa), and even at this very amplitude it is clear that $|p_1|/|p_0| << 1$ still holds.

In general we can identify acoustic pressure $p_1(\vec{x}, t)$ as our key acoustic variable being function of spatial position vector $\vec{x}$ and time t. Let's note for completeness that p_0 can also vary spatially, e.g., underwater p_0 is increasing by 10^5 Pa for every 10 m increase in depth, and in the atmosphere temporal changes occur with barametric pressure but on very large time scales. But both spatial and temporal changes in p_0 are on scales not comparable to the majority of acoustic processes and for our immediate purposes p_0 can be considered a constant in both space and time which, as we shall see, rarely enters into our analysis.

So with acoustic pressure p_1 representing the dynamic features of sound, there are also kinematic features, the most important of which is acoustic velocity $\vec{u}$, which we will can also call the fluid velocity. Acoustic pressure and velocity are the two most important acoustic variables, because with their combination we can identify acoustic energy and intensity (as shown later).

We won't use a subscript 1 in $\vec{u}$ because, unlike pressure, where there is always some background pressure p_0, without sound there is no background fluid velocity. Of course, winds in the atmosphere and currents underwater, characterized by a mean flow speed of $\vec{u_0}$ can enter into our analysis, but for our purposes of understanding linearity and the developing the wave equation, we are safe to assume $\vec{u_0} = 0$ (some caveats to this will be discussed later).

Acoustic velocity $\vec{u}$ is also often referred to as *particle velocity*. This term finds wide-spread, but not universal, use in journal articles and texts. Even though I'll confess having used it, too, in some of my journal articles, I prefer to avoid it now. The main reason is that *particle velocity*, and the similar one term *particle motion*, are often misunderstood. For example we'll show that $\vec{u}$ is linearly related to p_1 and other acoustic variables, so then why not use a description like "particle pressure"? This of course is completely inappropriate. In some classic text books, such Theoretical Acoustics, the authors never mention the word "particle" in their explanation on acoustics and the acoustic wave equation; because $\vec{u}$ is a continuous and smooth property of the medium, just like pressure, so describing it in the context of a particle has no real instructive purpose. Still, many good texts do use the term particle velocity but are also careful to provide some context about constitutes a fluid particle (or fluid parcel), e.g. a reasonably useful definition is from Fundamentals of Acoustics: "The term *particle of the fluid* means a volume element large enough to contain millions of molecules so that the fluid may be thought of as continuous medium yet small enough that all acoustic variables may be considered nearly constant throughout the volume element. "

Such context becomes more important when thinking about the Euler versus Lagrange descriptions of fluid flow. The Eulerian description means we are not studying a *particular* fluid parcel but instead the flow of fluid through some fixed control volume dV centered about a fixed laboratory reference frame. This is the most common approach in the study of dynamics and motion within the field of fluid dynamics, of which acoustics is branch of this field. We view dV as control volume and the Eulerian laboratory reference frame fits well with our notion of an acoustic sensor, e.g., microphone, or hydrophone, at a fixed position within the fluid medium. The alternative approach is a Lagrangian description, which defines the measurement process by following the motion of a particular parcel of fluid, say identified with colored dye. A good discussion on this topic is provided by Garrett (p. 425-430). To summarize, in ME 525 we will only use acoustic velocity (or sometimes I will slip in fluid velocity), to describe the small velocity of the fluid associated with the presence of a sound wave. We will use the Eulerian description, meaning the acoustic velocity is a function of the position $\vec{x}$ with components x, y, z and time t where it has been measured. Thus, as with acoustic pressure, we write $\vec{u}(\vec{x}, t)$.

First key relation: Conservation of momentum (Euler's equation)

In the following we develop the acoustic wave equation from two key relations, the linearized conservation of momentum and conversation of mass. First, let fluid density ρ be equal to ρ_0 the background mass density plus ρ_1 any change in density as result of passage of the sound wave, where $|\rho_1|/\rho_0 << 1$, (using $|\rho_1|$ as ρ_1 can sometimes be positive or negative, to represent a slight increase or decrease, respectively, in density. Let's reiterate how this works in the case of pressure and density:

$$p = p_0 + p_1 + p_2 + ...$$
$$\rho = \rho_0 + \rho_1 + \rho_2 + ...$$

where pressure p and density ρ are composed of successive approximations. Our interest ends really with the first order terms as in p_1, ρ_1 which are already very small with respect to the backgrounds p_0, ρ_0, with the second order terms p_2, ρ_2 even smaller yet. However these terms can come into play in study of *non linear acoustics*.

The rules for linearization are simple: products of small first order, small quantities (including temporal or spatial derivatives) are ignored. The small acoustic field quantities introduced thus far are p_1, ρ_1 and $\vec{u}$. We also assume that background quantities, as in p_0 and ρ_0 are constant (steady state), with spatial and temporal derivatives equal to zero. For example, apply this rule to $\frac{\partial(\rho u)}{\partial x}$ which equals $u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}$. The first term is eliminated given $\frac{\partial \rho_0}{\partial x} = 0$ and $u\frac{\partial \rho_1}{\partial x}$ is ignored as it represents a non-linear product of small quantities. Next convince yourself that the linearized version of second term is $\rho_0\frac{\partial u}{\partial x}$ which is truly linear in the small variable u, where here it is multiplied by a constant ρ_0.

In Fig. 3 let p stand for total pressure, equal to $p_0 + p_1$, where first term is the constant background pressure and the second term is due to the presence of sound. For this condition of a sound wave traveling only in the $x-$direction there exists a pressure differential $p(x)$ and $p(x + dx)$ that acts on both sides of a control volume dV of cross section S, centered about a fixed laboratory reference frame, with origin $x, y, z = 0$. The pressure difference gives a net force in the x direction:

$$S(p(x) - p(x + dx)) = -S\frac{\partial p_1}{\partial x}dx \tag{2}$$

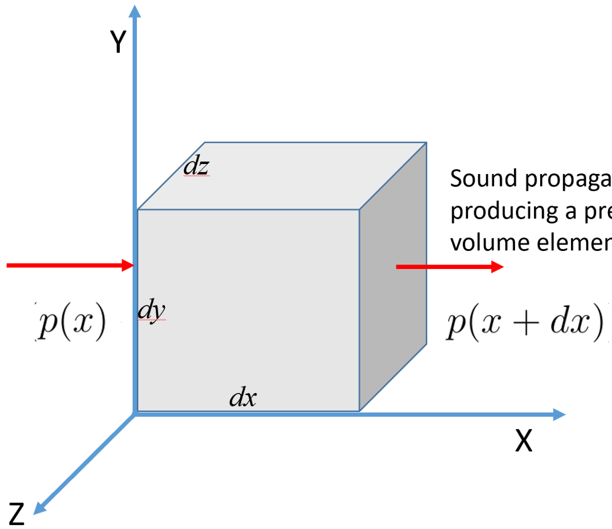

Figure 3: Sound propagation in either positive or negative x-direction produces a pressure differential $p(x)$ and $p(x + dx)$ that acts on both sides of a control volume dV of cross section S, centered about a fixed laboratory reference frame, with origin $x, y, z = 0$.

This net force translates to a "mass times acceleration" within the small volume:

$$-S\frac{\partial p_1}{\partial x}dx = Sdx\rho\frac{\partial u}{\partial t} \tag{3}$$

where the acoustic velocity vector $\vec{u}$ has components $[u, v, w]$ with u representing the x-component. Upon using our rules of linearization we find:

$$-\frac{\partial p_1}{\partial x} = \rho_0\frac{\partial u}{\partial t}. \tag{4}$$

The more general case where the sound in the y and z direction is also involved is:

$$-\nabla p_1 = \rho_0\frac{\partial \vec{u}}{\partial t} = \rho_0\vec{a} \tag{5}$$

This is the linearized form of Euler's equation most often used in acoustics. Keep it in mind, and we'll combine this later with conservation of mass to obtain the wave equation.

Summarizing, we have now have the following acoustic variables (also referred to as flow variables): p_1, ρ_1 and $\vec{u}$. We can also identify acoustic displacement $\vec{\xi}$, such that $\frac{\partial \vec{\xi}}{\partial t} = \vec{u}$, and acoustic acceleration $\vec{a}$, such that $\frac{\partial \vec{u}}{\partial t} = \vec{a}$.

Example measurement of acoustic pressure and velocity

This is enough for now so let's finish with a short example of acoustic measurements made

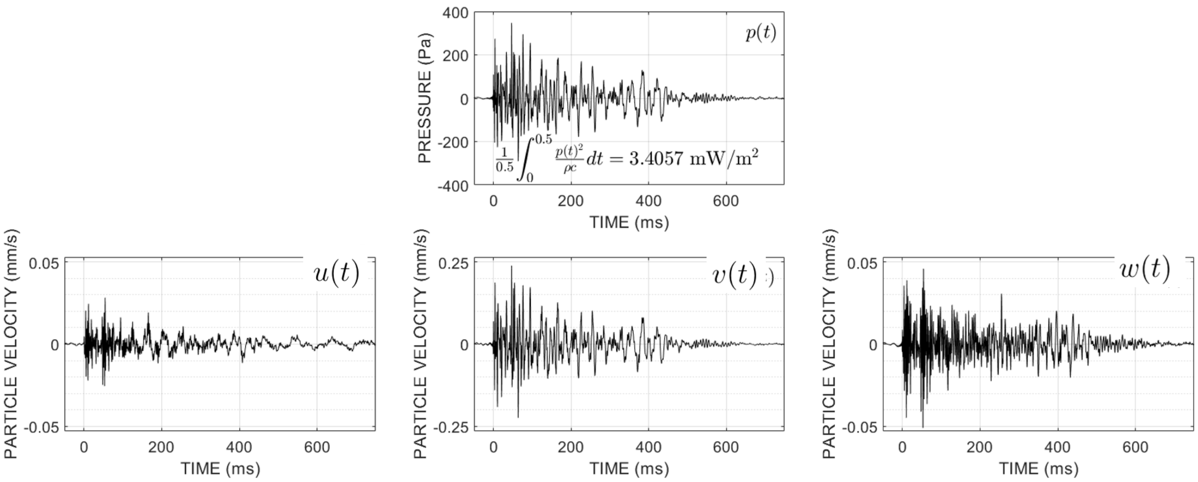

Figure 4: Acoustic pressure (top) and three components of acoustic velocity (bottom) of the underwater sound from an explosive source (31 g of TNT), made at range 10 km from the source in waters 75 m deep. Figure from Dahl and Dall'Osto (2019)

underwater, and a comment on the diverse field of acoustics.

Figure 4 is measurement of the pressure and Cartesian components of acoustic velocity, $\vec{u}$, given by $[u, v, w]$, of the underwater sound from an explosive source (31 g of TNT), made at range 10 km from the source in waters 75 m deep (as measured by my research team off New England.) The pressure (top row) reaches about 250 Pa in amplitude (quite high). Notice how it will oscillate around a mean value of 0, which is what we expect for sound. Putting this more into perspective, background noise might reach a peak value of about 1 Pa. However, does 250 Pa still satisfy our notion of $p_1/p_0 << 1$?

The three components of acoustic velocity are shown in the bottom row. Notice the y-component is the largest. This is because the explosive source bearing was aligned with y-axis of our sensor. The magnitudes of the $[u, v, w]$ are actually relatively large, e.g, with v being on the order of 0.25 mm/sec. Do these very small velocities also make sense? We'll know more as we learn about the relationship between acoustic pressure and fluid velocity; a key part of the relation is the quantity $\rho_0 c$ where c is the speed of sound in medium. For water c is about 1500 m/s and thus the acoustic mach number $|\vec{u}|/c$ is clearly small, being of order 10^{-6}, which we write as $O(10^{-6})$.

The diverse field of acoustics

Figure 5 is kind of an old-fashioned figure, representing the the ”wheel of acoustics” , a sketch made by R. Bruce Lindsey to summarize the diverse field of acoustics. Lindsey was a Professor of Physics at Brown University and (at the time) Editor-in-Chief of the Journal of the Acoustical Society of America, or JASA, the preeminent scientific journal for all things related to sound, published by the Acoustical Society of America, or simply the ”ASA”.

But the wheel is from 1965, and it gives only the most broad sketch of the field of acoustics as one might have viewed it 50 years ago. Since then we've seen noise canceling headphones, artificial speech and voice recognition (e.g., the Amazon ”smart speaker”), advances in simulation of sound in air and underwater, with the field ever changing.

Go to https://exploresound.org/explore-sound-home, produced by the ASA, to find out more about the very diverse field of Acoustics, as seen in 2021. Also, I encourage you visit the website of Acoustics Today (https://acousticstoday.org), a magazine produced by the ASA that you can download for free, including back issues, and also has an excellent on-line presentation. An issue in early 2020 has an article by my former PhD student, David Dall'Osto, on the finding of the Argentine submarine *San Juan* using advances in underwater acoustics. There's even an article on how plants can generate and respond to sound, and the new field of plant bioacoustics.

Returning now to the ”wheel”, I've highlighted in yellow my two broad areas of study, most of which involves underwater acoustics. Within this area there are many sub-specialties, as in underwater communication, sonar design, acoustical oceanography, etc. The goal of this course is to introduce acoustics through its various applications and sub-fields, such as medical ultrasound, sonar and underwater sound, noise studies and to introduce new perspectives on data analysis that can be applied to your particular field of research. So, I'm necessarily informed by my studies in underwater sound. But in the field of acoustics, techniques and concepts learned in one application, readily translate to other applications.

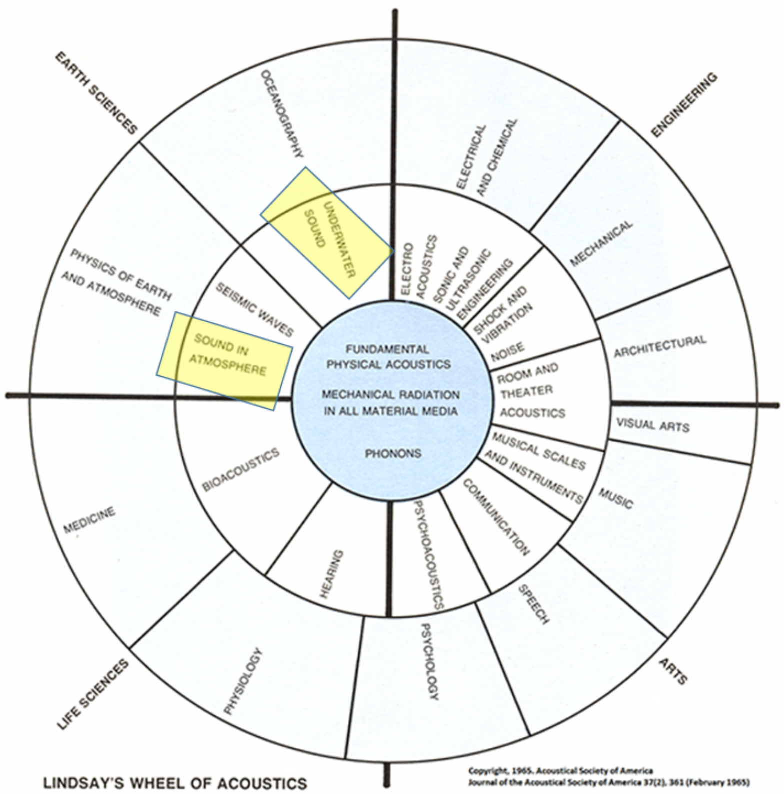

Figure 5: Lindsay's Wheel of Acoustics, sketch by R. Bruce Lindsey (former Editor-in-Chief of the Journal of the Acoustical Society of America) showing the fields of acoustics as they were viewed in 1965, starting with the four broad fields of Earth Sciences, Engineering, Life Sciences, and the Arts.

References

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L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, (John Wiley & Sons, New York, 1980)

S. Garrett *Understanding Acoustics, An Experimentalist's View of Acoustics and Vibration* (Springer ASA Press, 2017).

P. H. Dahl and D. R. Dall'Osto "Vector Acoustic Analysis of Time Separated Modal Arrivals From Explosive Sound Sources During the 2017 Seabed Characterization Experiment" IEEE *J. Oceanic Eng.* , 2019, DOI 10.1109/JOE.2019.2902500 (open source)

ME525 Applied Acoustics Lecture 2, Winter 2022

Peter H. Dahl

More on the general characteristics of sound

The fluid medium through which sound is propagating is characterized by a volume elasticity, the springlike property of the medium called *compressibility*, and often identified with symbol κ. Together with the fluid's mass characterized by density, you can roughly think of the fluid acting as a chain of masses and springs which allows a sound wave to propagate.

The speed of sound propagation in a fluid medium is $c = \sqrt{\frac{1}{\kappa\rho_0}}$. One can imagine a medium that is "more compressible" such as air, is defined by a larger κ and lower sound speed, than less compressible medium such as water which lower κ has a higher sound speed. Keep in mind that the κ for a medium, such as air or water, may vary somewhat depending on conditions. For air the main influence is temperature, and a useful empirical relation for the sound speed in air is $c \approx 20.05\sqrt{273.16 + T}$ such that the speed (c_{air}) is about 343 m/s at room temperature (using 20°). For water the main influences are temperature, salinity, and depth and for present purposes use a nominal value $c_{water} \approx 1500$ m/s. But we'll come back to this later, as the variation of sound with altitude in air, or depth in water, drives an important feature of sound propagation known as refraction.

Second key relation: Conservation of mass

In Lecture 1 we arrived at equation for the linearized conservation of momentum (Euler's equation). Here we derive the next key relation which is the linearized conservation of mass.

The equation of conservation of mass (also called continuity) gives a relationship between density ρ and acoustic velocity in the process of acoustic wave motion that necessarily involves a compression and expansion of the medium. We continue studying a small volume dV with cross section S set within a fixed, Eularian reference frame (Fig. 1). The mass flow into the elemental volume evaluated at x=0 is $\rho u S|_{x=0}$ having dimension mass per unit time or $\frac{M}{T}$, and the mass flow out of the elemental volume is $\rho u S|_{x=dx}$. For conservation of mass, the difference ought to equal the time rate of change of mass within the volume ($\rho S dx$) giving

$$\frac{\partial(\rho S dx)}{\partial t} = (\rho u S)|_{x=0} - (\rho u S)|_{x=dx} \tag{1}$$

the left side representing the time rate of change in density within the volume Sdx, from which we eliminate S and recast as formal, partial derivative

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0. \tag{2}$$

Now linearize Eq. (2), where $\frac{\partial(\rho u)}{\partial x}$ equals $u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}$. Eliminate the first term given $\frac{\partial \rho_0}{\partial x} = 0$ and $u\frac{\partial \rho_1}{\partial x}$ is ignored as it represents a non-linear product of small quantities. Convince yourself that what remains of the second term in $\frac{\partial(\rho u)}{\partial x}$ is $\rho_0\frac{\partial u}{\partial x}$ which is truly linear in the small variable, $\frac{\partial u}{\partial x}$, which here is multiplied by a constant ρ_0 considered a steady state, constant background value.

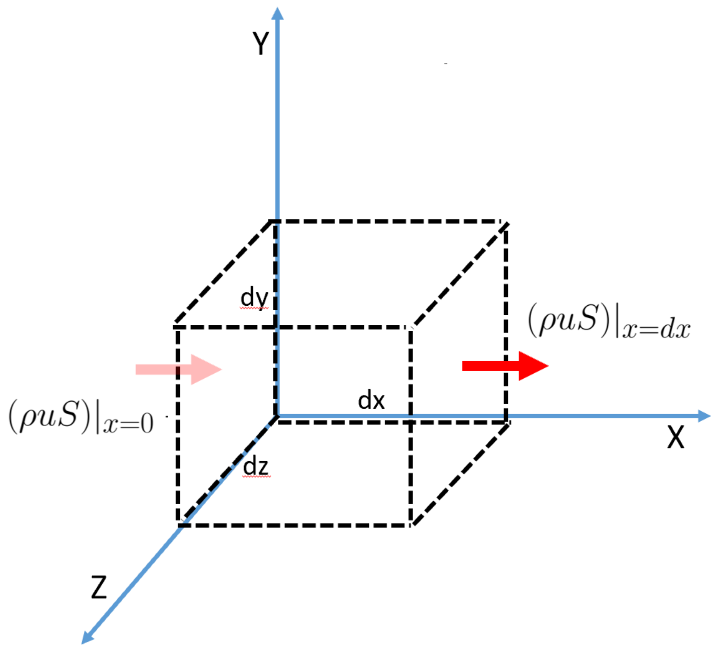

Figure 1: Fixed elemental volume of fluid $dV = dxdydz$, through which there is fluid motion characterized by the acoustic velocity in the x-direction equal to u, representing an in-flow (left) and out-flow (right) of mass. The volume dV is situated at fixed laboratory (Eulerian) reference frame.

We thus arrive at a 1-D linearized conservation of mass

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0 \tag{3}$$

and this is generalized to 3D by studying the mass flow through a cube, giving

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0. \tag{4}$$

It's worthwhile doing this another way (Fig. 2) while introducing another small acoustic variable: acoustic displacement vector ξ, with Cartesian components $[\xi_x, \xi_z, \xi_z]$. The acoustic displacement ξ_x is acting to "stretch" a piece fluid as result of the acoustic disturbance–indeed the piece of fluid can stretch, or compress, the degree to which is characterized by κ. Conservation of mass requires a balance between new volume $[dx + \xi_x(x + dx) - \xi_x(x)]S$ times the new density $\rho_0 + \rho_1 + ...$ equate to the original mass in the absence of sound $\rho_0 S dx$.

We again linearize and arrive at

$$\rho_1 + \rho_0 \frac{\partial \xi_x}{\partial x} = 0 \tag{5}$$

and for the process in 3D this relation becomes

$$\rho_1 + \rho_0 \nabla \cdot \vec{\xi} = 0. \tag{6}$$

Taking the time derivative of Eqs. (5) and (6) gives Eqs (3) and (4), respectively. Look carefully at the dimensions of the two terms Eqs. (5) and (6) – they must each have the dimension of density which ML^3, and means $\frac{\partial \xi_x}{\partial x}$ or 3D counterpart, $\nabla \cdot \vec{\xi}$ are non-dimensional.

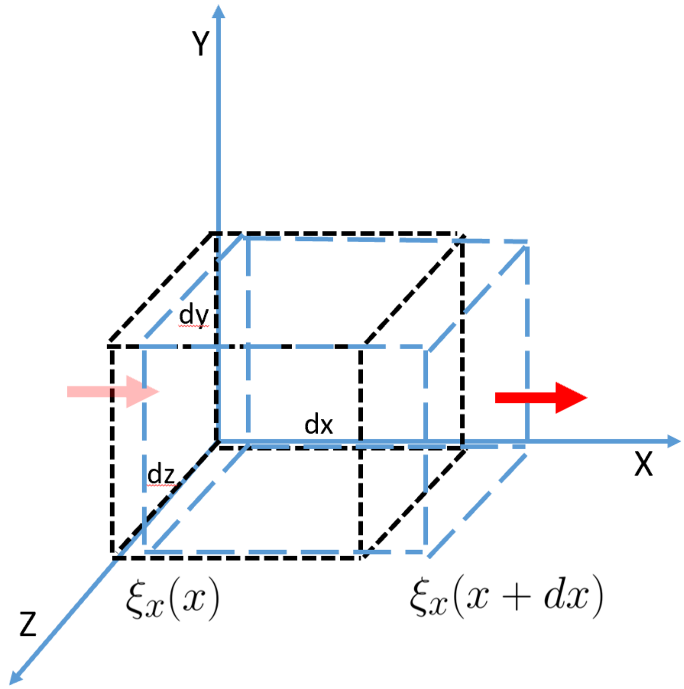

Figure 2: A volume $dV = dxdydz$, originally under equilibrium, is stretched (highly exaggerated in this figure) due to difference in displacements in the $x-$direction ξ_x at x and $x + dx$. Sound propagation direction is restricted to the $x-$axis (red arrow); whether sound is going left to right or vice versa is not relevant.

To interpret $\nabla \cdot \vec{\xi}$, think first about $\frac{\partial \xi_x}{\partial x}$, if $\frac{\partial \xi_x}{\partial x} > 0$ then the cube has been stretched a small amount along the $x-$dimension (Fig. 2) The stretched volume (now larger) produces a small change in density ρ_1 that is negative. The small, positive, and non-dimensional $\frac{\partial \xi_x}{\partial x}$ multiplies the background density ρ_0 to balance this according to Eq.(5). The opposite happens for $\frac{\partial \xi_x}{\partial x} < 0$ producing a small, positive ρ_1.

The expression $\nabla \cdot \vec{\xi}$ is divergence of displacement,

$$\nabla \cdot \vec{\xi} = \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \tag{7}$$

which is known as *cubic dilation*.

We have now encountered two applications of the differential operator ∇. The first ∇p_1 from the linearized Euler equation [Eq(5) of Lecture 1], takes the gradient of the scalar acoustic pressure p_1 resulting in a vector

$$\nabla p_1 = [\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}]p_1 \tag{8}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in the x, y, z directions. The second, $\nabla \cdot \vec{u}$ or $\nabla \cdot \vec{\xi}$, operates in similar manner but on a vector quantity, resulting in scalar. The dot-product relation insures components of the vector are matched up with spatial derivatives along corresponding axes, as in Eq.(7).

A practical application

The linearized form of Euler's equation from Lecture 1 states that we can take a pressure gradient, divide by ρ_0 and arrive at an estimate of acoustic particle acceleration $\vec{a}$. This has much practical significance: in acoustics we will take the finite difference between pressure measurements made from two closely-spaced hydrophones (underwater) or microphones (air). We such data we find the pressure gradient in the direction along the axis of the two sensors and from which acoustic particle acceleration along that same axis is estimated.

This equation is also very important in noise control. If we can measure $\vec{a}$ or $\vec{u}$ then we also know more about the direction of sound, and the propagation of sound energy (more on that later). For example, commercial sound intensity probe (Fig. 3) is used to measure and localize sound emissions from an egg beater. In addition to small appliances, you can well imagine car and airplane manufactures doing this to localize and reduce sound sources. The probe works by having two microphones precisely held in place with small separation (Δr in the figure). A pressure gradient is measured in the r direction (here called $\frac{\partial p}{\partial r}$), and since particle velocity is desired instead of $\vec{a}$ this quantity is time integrated.

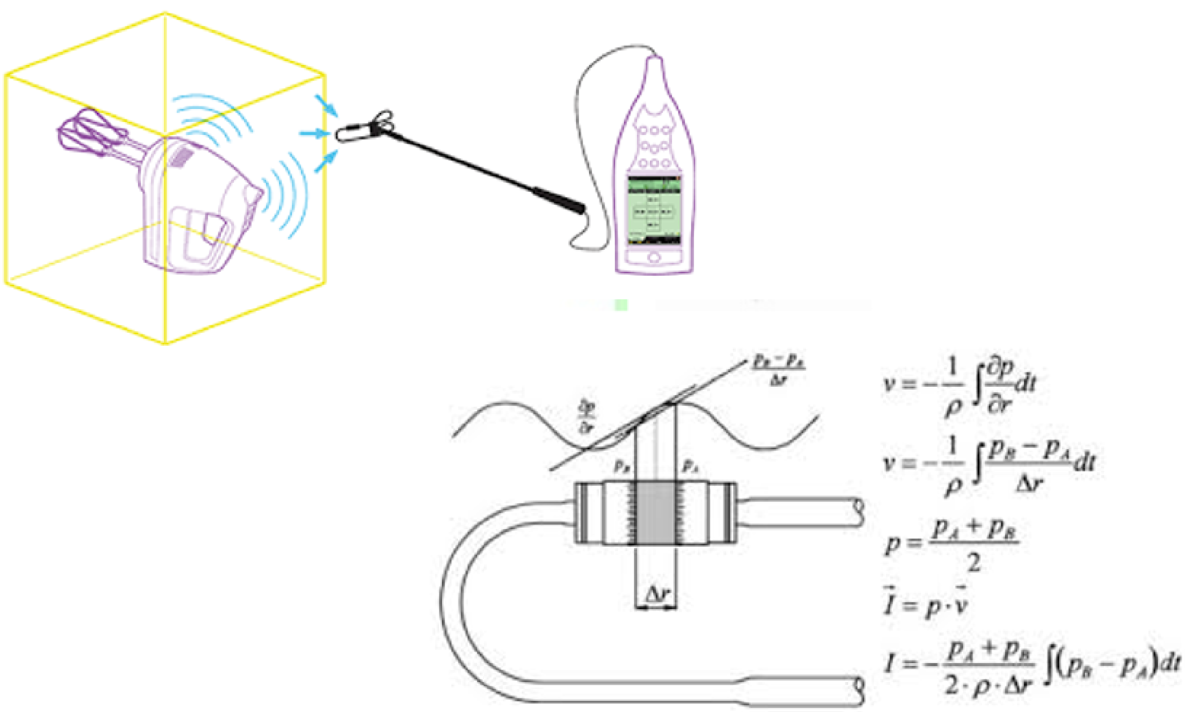

Figure 3: In upper figure an intensity probe is used to "see" sound from an egg-beater coming from different directions in order to localize various sound sources for the purpose of more effective noise reduction. In the lower figure we see the basic mathematics of an intensity probe. This is based on a finite difference approximation for the pressure gradient using pressure measurements p_A and p_B separated by Δr

Summarizing Lectures 1 and 2, the key points are:

- Linear acoustics: different sounds add without distortion!
- Take advantage of successive approximation series, e.g., for fluid density: $\rho = \rho_0 + \rho_1 + \rho_2 + \ldots$, where $\rho_1/\rho_0 << 1$. The steady-state background (equilibrium) density = ρ_0.

- For the linearization drill, products of two small quantities are ignored: e.g., $u\frac{\partial \rho_1}{\partial x} << \rho_0 \frac{\partial u}{\partial x}$
- Acoustic data can range over several orders of magnitudes. Consider plotting in log scale– don't get lazy with a linear scale or you can loose the big picture!
- We arrived at a linearized conservation of momentum (Euler's equation): $\frac{-\nabla p_1}{\rho_0} = \vec{a}$ which is a vector equation,
- and the linearized conservation of mass (continuity): $\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0$, which is a scalar equation.
- Pay attention to dimensions! Do these equations make sense in terms of fundamental dimensions (mass, length, and time or M, L, T)? There is no cheaper insurance than good practice in dimensional analysis.

Lecture 2 Appendix: The divergence theorem

The purpose of this appendix is to give you a feel for how to address conservation principles in a more arbitrary volumes using some techniques in vector calculus. If you've taken ME 564 you are likely familiar with some of these techniques, but if not–then try to follow the meaning of these equations anyway using dimensional analysis. We don't really use such techniques in this course–so again this is just for your background–however I will provide an interesting example of their use later in the quarter.

For an arbitrary vector field $\vec{A}$ the divergence theorem of Gauss allows us to write

$$\int_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot \vec{A} dV \tag{9}$$

where the volume V is bounded by the surface S.

Conservation of mass situation for an arbitrary volume subject to an acoustic disturbance is as follows:

$$\frac{\partial}{\partial t} \int_V \rho dV + \int_S \rho \vec{u} \cdot d\vec{S} = 0 \tag{10}$$

The first term is the time rate of change of mass inside volume V which must be balanced by the new mass inflow/outflow through the boundary S. The vector $d\vec{S}$ has magnitude dS and direction outward normal to the arbitrary surface S.

Apply next the divergence theorem to get everything into one volume (including bringing the time derivative inside the integral)

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot \rho \vec{u} dV = 0 \tag{11}$$

This situation must apply to an arbitrary volume, and thus the two volume integrals can be combined, and the integrand set to 0, giving

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0 \tag{12}$$

Again linearize to arrive at:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0 \tag{13}$$

References

P.M. Morse and K. U. Ingard, *Theoretical Acoustics*, (Princeton University Press, 1984). See Sec. 6.

S. Garrett *Understanding Acoustics, An Experimentalist's View of Acoustics and Vibration* (Springer ASA Press, 2017). See Sec. 8.

ME525 Applied Acoustics Lecture 3, Winter 2022

Peter H. Dahl

The acoustic Equation of State

The equation state provides a functional relationship among thermodynamic variables density, pressure and temperature, such as $\rho = \rho(P, T)$ or $P = P(\rho, T)$, as in the ideal gas law from chemistry $PV = nRT$ where n in the number of moles, R is the gas constant, and P, V and T are pressure, volume and temperature.

For our purposes start with a generic equation of state: $p = p(\rho)$, and do a Taylor series expansion around the background density ρ_0 as follows:

$$p = p(\rho_0) + \frac{dp}{d\rho}|_{\rho_0}(\rho - \rho_0) + \frac{d^2p}{d\rho^2}|_{\rho_0}\frac{(\rho - \rho_0)^2}{2} + ... \tag{1}$$

Identify the first term as background pressure p_0, identify $(\rho - \rho_0)$ as ρ_1, and associate a new constant, the square of sound speed c^2, with $\frac{dp}{d\rho}|_{\rho_0}$. We can stop at this first, linear term, and thus evidently

$$p_1 = c^2\rho_1 \tag{2}$$

which represents our key relation between the two acoustic flow variables p_1 and ρ_1, and which is the *acoustic equation of state*.

Note that Eq.(1) follows from Blackstock (p. 32), but not exactly, as there it is formulated slightly differently to identify coefficients $A, B, C..$ in successive terms starting with the second term in Eq.(1), but yielding same result. In study of non-linear acoustic effects, such as in the fields of industrial or medical ultrasound, one often come across the phrase "B-over-A", representing an important parameter quantifying non-linear effects. Blackstock also refers to a "small signal speed" c_0, for the important relation $p_1 = c_0^2\rho_1$, to leave room for discussing the non-linear influences embodied by the higher order terms in the above series expansion.

However, stopping at the first term satisfies most engineering applications and officially, you may now associate change in pressure associated with sound, p_1 with corresponding change in density ρ_1 via a constant c^2 without subscript. This is a nice example of how in linear acoustics, two small acoustic variables p_1, ρ_1 are related through a constant; in this case it is c^2. It also provides a convenient pathway for how we can measure very difficult-to-observe quantities, like ρ_1–how might one do that? First measure p_1 using a microphone in air or hydrophone underwater, then divide by c^2 which we already know (or can look up in tables).

For example, take an acoustic pressure p_1 of ~ 2 Pa, perhaps representing the sound environment were you to sit on the hood of car with big "chevy" V8 engine while the engine was throttled up to high RPM (you people probably don't even remember a "chevy", or even a V8). Using c of 343 m/s, means ρ_1 is only about 1.7 10^{-5} kg/m^3, which is a pretty small change from the background density of air given this very loud sound.

In Lecture 2, we identified κ as compressibility, such that $c = \sqrt{\frac{1}{\kappa\rho_0}}$, so clearly the acoustic equation of state addresses the property compressibility in the medium. Another relation is the bulk modulus K of the fluid where

$$K = \rho_0 \frac{dp}{d\rho}|_{\rho_0} \tag{3}$$

therefore $K = \rho_0 c^2$ or $c = \sqrt{\frac{K}{\rho_0}}$; it thus follows that $K = 1/\kappa$. (As with κ, it is likely you will come across different notations for the bulk modulus different texts.)

Now, K can be interpreted as the ratio of the pressure increase to fractional volume decrease, caused by increased pressure, thus K has the same dimension as pressure. For example, K for seawater is about 2.3 GPa, and for air it is about 116 kPa. Let's revisit Eq.(6) from Lecture 2, and see if you can now recast it as follows:

$$p_1 = -K\nabla \cdot \vec{\xi} \tag{4}$$

Assume next a small increase in pressure $p_1 = 1$Pa for cubic meter of water caused by sound, given K the relative change in this volume is $1/K$ or about $\sim$ 4 10^{-10}. Repeat this for a cubic meter of air and the same pressure produces a relative change of $\sim$ 9 10^{-6}, or a more than 10000 fold increase in expansion over that of water.

The Wave Equation

An equation for acoustic wave propagation (wave equation) now falls into place given that we have a workable acoustic equation of state in the form of Eq. (2). Let's first summarize the two key parts that combined to form the wave equation:
The linearized conservation of mass:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0 \tag{5}$$

The linearized Euler's equation (conservation of momentum):

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p_1 = 0 \tag{6}$$

1. Take the divergence ($\nabla\cdot$) of Eq. (6) to get:

$$\rho_0 \frac{\partial}{\partial t} \nabla \cdot \vec{u} + \nabla^2 p_1 = 0 \tag{7}$$

2. Take the time derivative ($\frac{\partial}{\partial t}$) of Eq. (5) to get:

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \nabla \cdot \vec{u} = 0 \tag{8}$$

now observe:

$$\nabla^2 p_1 - \frac{\partial^2 \rho_1}{\partial t^2} = 0 \tag{9}$$

3. Finally apply the acoustic equation of state to eliminate ρ_1 from the above to arrive at a wave equation for acoustic pressure, to yield

$$\nabla^2 p_1 - \frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} = 0 \tag{10}$$

and 1-D counter-part if sound propagation is limited to one direction, say along the x-direction:

$$\frac{\partial^2 p_1}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} = 0. \tag{11}$$

Observe that wave equation uniquely relates the rates of change of p_1 with respect to position and time with the latter scaled by $1/c^2$. The same equation applies to the other acoustic variables, for example, convince yourself that a 1-D wave equation for acoustic displacement in the x-direction ξ_x is

$$\frac{\partial^2 \xi_x}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \xi_x}{\partial t^2} = 0 \tag{12}$$

Some simple interpretations of the acoustic wave equation

Solutions to the wave equation depend on the boundary conditions e.g., if sound is to be studied in open air, within a closed room, or underwater, these all require different boundary conditions to represent walls or the air-water interface, etc. Additionally, the right hand side of Eqs. (10-12) currently equal 0, meaning the equation is homogeneous. Adding sound sources, changes this, and we postpone discussion on this issue for later; for now let us see how basic solutions go, and to what extent they describe what we already perceive in our own acoustic world.

A basic solution to the wave equation is in the form of $p = F(x \pm ct)$, (similarly for ξ, u). The $\pm ct$ accommodates a wave traveling backwards or forwards in the x-direction. To fix ideas, assume for now $p = F(x - ct)$, and this will have the meaning of a sound wave traveling in the positive x direction.

The arbitrary function F (Fig. 1) might describe acoustic pressure at time t_1 at a specific position x_1, or $F(x_1 - ct_1)$ associated with some acoustic disturbance in the fluid. After an increment of time Δt the function F will have moved to position $x = x_1 + \Delta x$. The argument of F remains the same

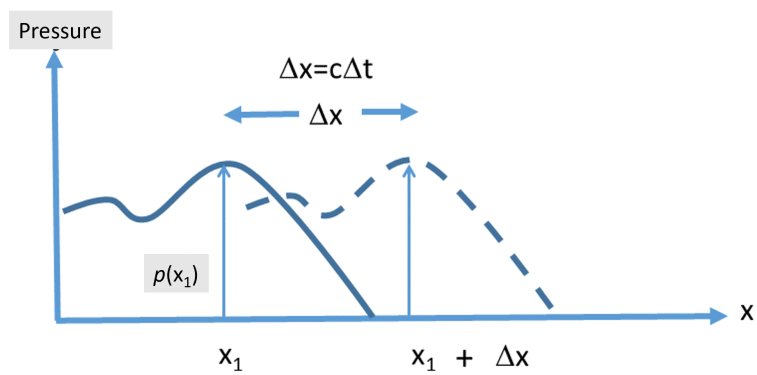

Figure 1: A simple shape function F describing pressure p as function of x. Solid line shows shape function F at time t_1 and dashed line at time at time $t_1 + \Delta t$.

however, as in $x_1 - ct_1$ equals $(x_1 + \Delta x) - c(t_1 + \Delta t)$, because the wave has traveled $\Delta x = c\Delta t$. In other words, after a short time Δt the same value of pressure will repeat further down the x-axis, or the whole function F, which we might call a shape function, moves from left to right, with same value $p_1(x_1)$ occurring at position $x_1 + \Delta x$. The velocity of travel of the entire shape function F is $\Delta x/\Delta t = c$.

Though this illustration is a bit simplified, it still shows how the shape-preserving property of the wave equation for small-amplitude disturbances (i.e. ordinary sound waves) which has profound implications: were this not to be the case, then those sitting a few rows ahead of you in a theater would have a very different sound experience, because the basic shape of the waveform would very with distance.

There for sure is a distance-effect relating to the *amplitude* of sound, e.g. the spreading of sound with range from source, which we discuss later, but it is this shape-preserving property that is key to the propagation of sound in the linear regime. You should check out some of the sound wave visualization demonstrations on the websites I've posted on our ME 525 website.

Closely related to the shape-preserving property is the concept of linearity and superposition. Let now an acoustic pressure field consist of the sum of N sources, as in $p_1 = \sum_i^N F_i(x - ct)$ where the shape function of the i^{th} source is $F_i(x - ct)$. The total sound field is a *linear superposition* of the individual sources, for which the final result can be very complex but within that result individual sources contribute (and can be heard) without distortion caused by other sources, e.g., as in the recording discussed in Lecture 1.

Were linearity not to be the case the net result of sound symphony orchestra would be a cacophony. If you want to observe linear superposition in action, look closely at two gravity wave fields coming from different directions on the sea surface. If these eventually meet then at that spot there is momentary change in the sea surface shape, after which the two wave trains pass each as if nothing happened. Finally, note that *any* function of the form $F = f(x - ct)$ represents a solution to wave equation, as in $\xi = A\sin(a(x - ct))$, or $\xi = A\ln(a(x - ct))$.

Simple-harmonic time dependence

One form of $p_1 = F(x - ct)$ that we'll find much use for is the case representing a single acoustic frequency f, or harmonic solution. It is very convenient to employ harmonic solutions and represent a single frequency f, in a complex exponential such as $p_1 = Ae^{i(kx-\omega t)}$ where ω equals $2\pi f$ and where the wavenumber k equals $\frac{\omega}{c}$. Plug this into Eq.(11) and it better work.

Note that in my research, and in this course, I prefer to express complex harmonic dependence as $e^{-i\omega t}$. Many books and technical articles (most often in electrical engineering) use $e^{+i\omega t}$. There is really no difference, particularly in the study of simple systems where time t is the only independent variable.

However, we also need to study how the field varies with position and time, as in the $e^{i(kx-\omega t)}$, and it is somewhat more intuitive to have the position part vary as e^{ikx} with positive increasing position (here along the x-axis), rather than write $p_1 = Ae^{-i(kx-\omega t)}$. Otherwise, does this difference matter? No, so long as one remains consistent and does not mix the two conventions.

More on complex exponential notation

We use complex exponential notation because manipulations and calculations are *soo* much more easily carried out. But what does it mean to write $p_1 = Ae^{i(kx-\omega t)}$ when this is supposed to represent a measurement? It means that we take the *real part*. Let's suppose A is also complex, say equal to $A = |A|e^{i\phi_A}$ where ϕ_A is the phase of A, then the *real part* of $Ae^{i(kx-\omega t)}$ equals $|A|\cos(kx - \omega t + \phi_A)$.

Suppose you measure a sound field at some fixed point in space x_o that can be modeled by $p_1 = Ae^{i(kx_o-\omega t)}$, where f equals 2 Hz (this might be infrasound), and A is a complex amplitude (assume for simplicity $|A| = 1$). A plot (Fig. 2) of the measured sound is obtained by taking the real part of this model. Two examples are shown, for the black line $A = e^{i\pi/12}$ and for the magenta line $A = e^{i\pi/3}$. These measurements thus represent two acoustic waves of identical magnitude, and frequency, but different phases. So using complex representation allows for a more convenient representation of phase.

The big tip: probably one of the most handy things about using complex exponential notation for simple-harmonic time dependence, i.e. the time dependence is $e^(\pm i\omega t)$, is how easy it is to compute rms values. For example, assume a harmonic spherical wave, $p(r,t) = \frac{A}{r}e^{ikr-i\omega t}$ (we discuss spherical waves next week) which has real part representing some measured pressure $p_m(r,t)$

$$p_m(r,t) = \frac{|A|}{r}\cos(kr - \omega t + \phi_A), \tag{13}$$

The mean-square is defined as the time integral over one period T

$$\frac{1}{T}\int_0^T p_m^2(r,t)dt = \frac{|A|^2}{r^2}\frac{1}{T}\int_0^T \cos^2(kr-\omega t+\phi_A)dt \tag{14}$$

giving the rms pressure as $p_{rms} = \frac{1}{\sqrt{2}}\frac{|A|}{r}$. Easier still, is whenever we require mean-square values of complex harmonic quantities, then we are interested in computing $pp^\star$ over this same averaging period T (often we denote as $\langle p(t)p(t)^\star\rangle$), where $p^\star$ is the conjugate, where in this example $pp^\star = \frac{|A|^2}{r^2}$. The averaging time integral is now seen by inspection to yield the same result.

In summary for *any* complex harmonic function of the form $x(t) = Be^{\pm i\omega t}$ and B can also be both complex and also have spatial variation,

- the mean square value $\langle x^2(t)\rangle$ equals $\frac{1}{2}|B|^2$,
- the rms value equals $\frac{1}{\sqrt{2}}|B|$

Exploiting this fact will help make complex life much easier.

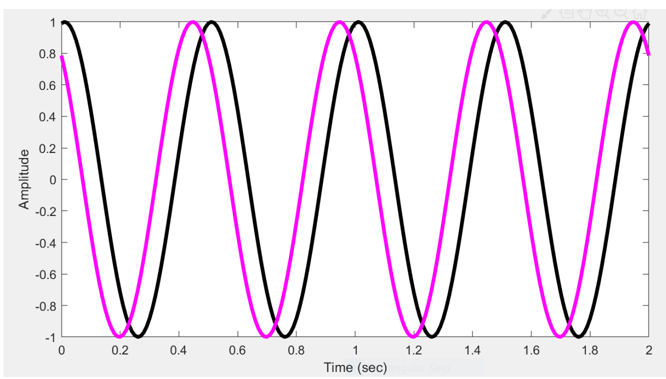

Figure 2: Two sinusoidal varying sound waves of frequency 2 Hz, measured as at the same position x_o. Data can be represented as $p_1 = Ae^{i(kx_o-\omega t)}$, and the magenta data has complex amplitude $A = e^{i\pi/3}$, the black data has complex amplitude $A = e^{i\pi/12}$

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K. U. Ingard, *Notes on Acoustics*, (Infinity Science Press, Hingham MA, 2008).