CHAPTER 5

STRUCTURAL VIBRATIONS

5.1 Structure-Borne Sound

In the previous two chapters we dealt with a number of sources that radiate sound directly into the fluid medium. However, many noise sources in marine systems are not in direct contact with the fluid. Vibrations generated by such sources are transmitted by structures to radiating surfaces. The transmission of structural vibrations and radiation of sound by such vibrations constitute the subject of *structure-borne sound*. This field was originally developed in Germany in the 1940's where it was applied both to submarine noise reduction and to building acoustics. An early exposition by L. Cremer (1950) is still a classic. Rapid progress in understanding the role of structures in both architectural and marine acoustics was made in the 1950's and 1960's, and the field can now be considered to have matured. Several recent books devoted exclusively to this subject are included in the references at the end of this chapter.

Most structures can be classified as either beam-like or plate-like. Beams are structures having only one dimension long compared to a vibrational wavelength, while plates have two dimensions that are relatively large. While it is somewhat of an oversimplification, it is nevertheless generally true that the primary role of beam structures is transmission of vibrations; plates, on the other hand, are usually in contact with a fluid medium and so are principal radiators of structure-borne sound. The present chapter deals with structural vibrations of beams and beam-like structures, while Chapter 6 covers sound radiation from plate vibrations.

Structures can experience a number of types of vibrations, as discussed in Section 5.2. However, the dominant mode of transmission of vibrational energy is by flexural (bending) vibrations and these are also the most efficient radiators of sound. It was recognition of the central role of flexural vibrations that has made possible the tremendous progress in this field of the past 40 years. Chapters 5 and 6 deal almost exclusively with flexural vibrations and their radiation.

In addition to the role of flexural motions in transmitting vibrations from machines to the hull where they can radiate sound into the water, they are also a dominant low-frequency vibrational mode of ship structures. The final two sections of the present chapter deal with beams immersed in fluids and with the calculation of low-frequency bending vibrations of ship hulls, called *whipping* modes in Section 4.11.

5.2 Wave Motions in Solids

As an introduction to the treatment of structural vibrations, it is useful first to define the various types of sonic vibrations that can occur in solids. Some of these vibrations occur only within the body of the solid, while others involve its surfaces. Solids of various shapes can be classed according to the ratios of various dimensions to each other and to a wavelength. Thus, as mentioned in Section 5.1, *rods*, *bars* and *beams* are structures having only one dimension large compared to a wavelength, while *plates* have two large dimensions, and all of the dimensions of *bulk solids* are relatively large.

In Section 2.2, the wave equation was derived for acoustic disturbances in liquids. One of the assumptions made as a part of the derivation was: "the fluid cannot withstand static shear stresses, in the manner of a solid." Obviously, then, a most important distinction between solids and liquids is the ability of a solid to withstand static shear stresses. Because of this property, wave motions in solids are considerably more complex than those in fluids. Not only do solids transmit compressional waves similar to those in fluids, but also they sustain shear waves, flexural (bending) waves and various combinations of compressional, shear and flexural waves, as well as surface waves.

The solids of interest in structure-borne sound are for the most part homogeneous and isotropic. The vibrations are of small amplitude and so may be treated as linear, or Hookesian. Thus any deformations, or strains, are directly proportional to the stresses causing them.

Longitudinal Waves in Bars

One form of wave motion in solids is that which arises if one strikes, or otherwise excites, one end of a thin rod or bar. A longitudinal compressional wave is set up in the bar which travels at a speed, c_ℓ, given by

$$c_\ell = \sqrt{\frac{Y}{\rho}}, \tag{5.1}$$

where Y is *Young's modulus*, defined as the stress required to produce unit strain, and ρ is the density. Most metals commonly used in structures have longitudinal wave speeds between 4900 and 5200 m/sec.

Shear Waves

Another form of wave motion that can exist in a solid is associated with twisting, or torsional, motions which occur in a plane perpendicular to the direction of propagation of the wave. Such waves depend on the ability of a solid to sustain shear. They propagate at a speed, c_s, which depends on the shear modulus of the solid,

$$c_s = \sqrt{\frac{G}{\rho}}. \tag{5.2}$$

The *shear modulus*, G, is invariably less than half Young's modulus, so shear waves travel more slowly than do longitudinal waves.

Compressional Waves in Bulk Solids

When longitudinal motions take place in a thin rod, there are associated changes in the diameter and cross-sectional area which reduce the relative volume change. These lateral changes play an important role in the wave process, making it easier for waves to propagate. When all the dimensions of a solid body are large compared to a wavelength, the lateral motions cannot occur in the same way as in a rod, and the medium is effectively less compressible. Defining the *bulk modulus*, B, as the ratio of the hydrostatic pressure to the fractional change in volume or density, as in Eq. 2.53, irrotational compressional waves are found to travel in the volume of a large solid at a speed given by

$$c_B = \sqrt{\frac{B + 4/3\,G}{\rho}}\,, \tag{5.3}$$

which expression reduces to that for a liquid when the shear modulus, G, approaches zero.

Poisson's Ratio

The quantity that measures the lateral constrictions of a rod experiencing longitudinal vibrations is called *Poisson's ratio*. It is the ratio of the relative change in the diameter of the rod to the relative change in length:

$$\sigma \equiv - \frac{L}{D}\frac{dD}{dL}\,. \tag{5.4}$$

The relative change in area is 2σ, while that of the volume is $1 - 2\sigma$. It follows that virtually incompressible solids, such as rubber, have values of Poisson's ratio close to 0.5. Values for most metals are between 0.27 and 0.35.

The three elastic moduli, Y, G and B, are related to each other through Poisson's ratio by

$$\frac{G}{Y} = \frac{1}{2(1 + \sigma)} \tag{5.5}$$

and

$$\frac{B}{Y} = \frac{1}{3(1 - 2\sigma)}\,. \tag{5.6}$$

It follows that the wave speeds are related by

$$c_s = c_\ell \sqrt{\frac{1}{2(1 + \sigma)}} \tag{5.7}$$

and

$$c_B = c_\ell \sqrt{\frac{1 - \sigma}{(1 + \sigma)(1 - 2\sigma)}}\,. \tag{5.8}$$

In the limit, for an incompressible solid, the shear wave speed equals about 3/5 of the longitudinal wave speed; bulk waves cannot be sustained at all, since c_B approaches infinity. These relations for c_s and c_B relative to c_ℓ are plotted in Fig. 5.1.

Longitudinal Waves in Plates

Plates are solids that are large compared to a wavelength in two dimensions and smaller than a

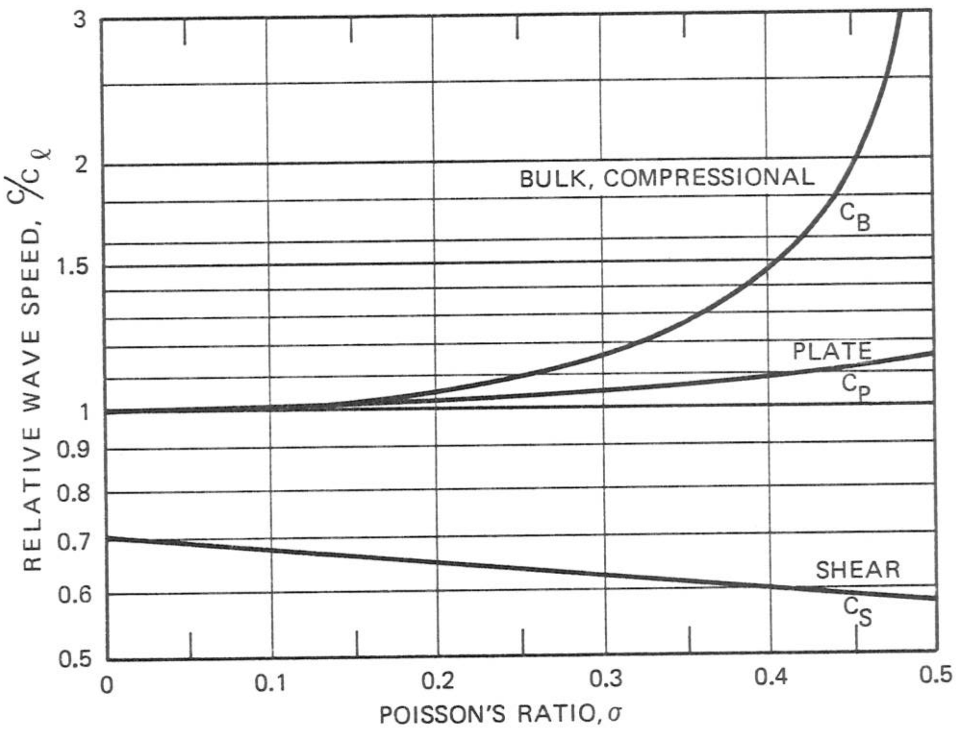

Fig. 5.1. Wave Speeds in Solids as Functions of Poisson's Ratio

wavelength in one dimension. Longitudinal waves travel in plates at a speed, c_p, which is slightly larger than the speed in bars and rods:

$$c_p = c_\ell \sqrt{\frac{1}{1 - \sigma^2}} = c_s \sqrt{\frac{2}{1 - \sigma}} . \qquad (5.9)$$

As shown in Fig. 5.1, this ratio varies from about 1.05 for typical metals to 1.15 for rubber-like, almost incompressible solids.

Surface Waves

Waves can propagate on the surface of a thick solid in much the same manner as surface waves do on the ocean. Such surface waves, which decay exponentially toward the interior, are called *Rayleigh waves* and are of considerable importance at very high frequencies, especially in ultrasonics applications. Smaller waves found on plates are termed *Lamb waves* and are also important in ultrasonics.

When a longitudinal wave propagates in a plate, the surfaces experience up and down motions associated with the Poisson effect. These surface waves can also radiate sound. However, most plate structural vibrations occur as bending motions, and radiation associated with longitudinal plate waves is usually less important.

Flexural (Bending) Waves

Beams and plates often experience wave motions in which one surface is experiencing stretching at the same time that the opposite surface is experiencing contraction. As shown in Fig. 5.2, the result of this combination is that the center of the beam or plate oscillates about the

Fig. 5.2. Flexural Wave in a Beam or Plate

rest plane. In this type of wave motion, the structure flexes, or bends. For small amplitude vibrations, the amount of stretching and contraction is a linear function of the distance from a plane, termed the *neutral plane*, which experiences neither, as shown in Fig. 5.3.

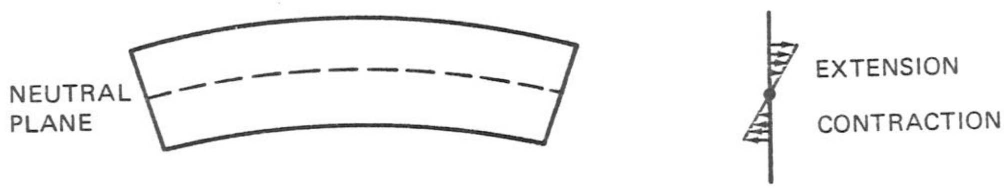

Fig. 5.3. Stretching and Contraction of a Section of a Bending Beam

Examples of flexural vibrations from everyday experience including tuning forks, tall buildings swaying in the wind, bending vibrations of aircraft wings, and the quiver of an arrow striking a target. They are important because they are readily excited. A given force will generally cause much larger flexural amplitudes than any other type of vibration.

Cremer made a significant contribution to architectural acoustics in recognizing the overriding importance of bending (flexural) vibrations for both the transmission of vibrations in structures and the radiation of sound. Since flexural waves also play the dominant role in vibration of and radiation of sound by ship hulls and other structures, the remainder of this chapter is devoted to a detailed exposition of their properties.

5.3 Beam Bending Equations

Differential equations relating to bending vibrations of beams can be derived by considering the forces and moments acting on a differential element together with the motions resulting from such action. Figure 5.4 shows such a beam element of length dx. The displacement of the neutral plane from an arbitrary reference is represented by w. The cross-sectional plane is shown rotated by an angle θ from the normal. The element experiences shear, hence the angle of rotation of the cross-sectional plane is greater than that of the neutral plane. A fiber element is shown at a distance z from the neutral plane. The longitudinal strain, or extension, of this fiber is represented by $d\xi$. The width of the element is represented by $b(z)$.

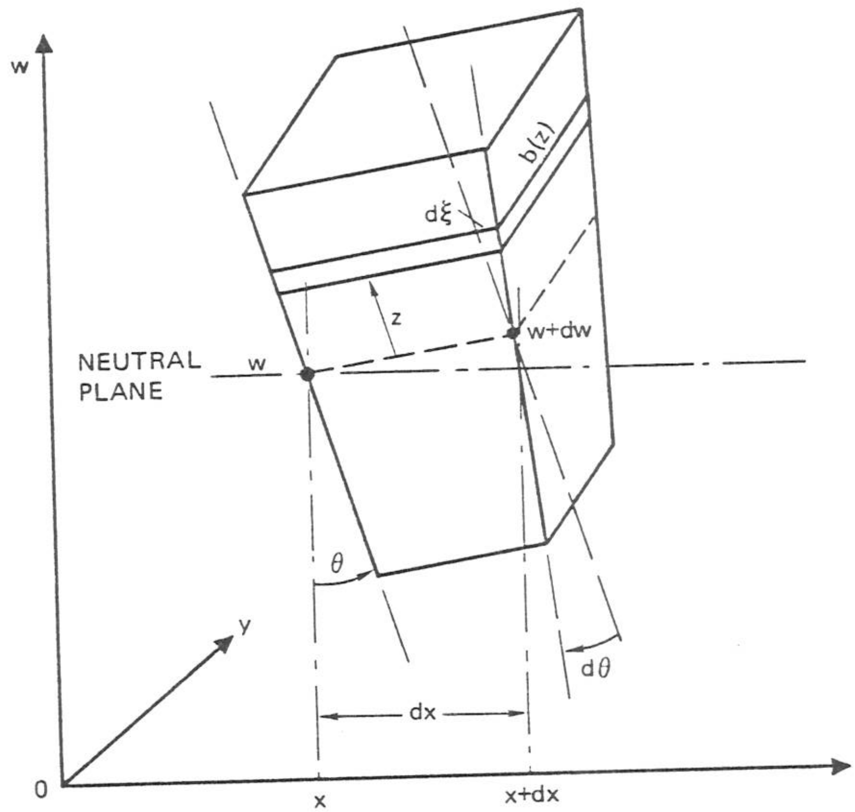

Fig. 5.4. Element of a Bending Beam

Forces and Moments

Figure 5.5 shows the forces and moments experienced by an element in bending. Although the net extensional force on the element, F_x, is zero, each fiber experiences a force proportional to its extension and given by

$$dF_x = -Yb\,d\xi = -Yb\frac{\partial \xi}{\partial x}dx\,, \tag{5.10}$$

where Y is Young's modulus and b is the width of the element, as previously defined. From Fig. 5.4 it is clear that the extension of each fiber is proportional to its distance from the neutral plane and is also a function of the curvature of the element. From geometric considerations, it

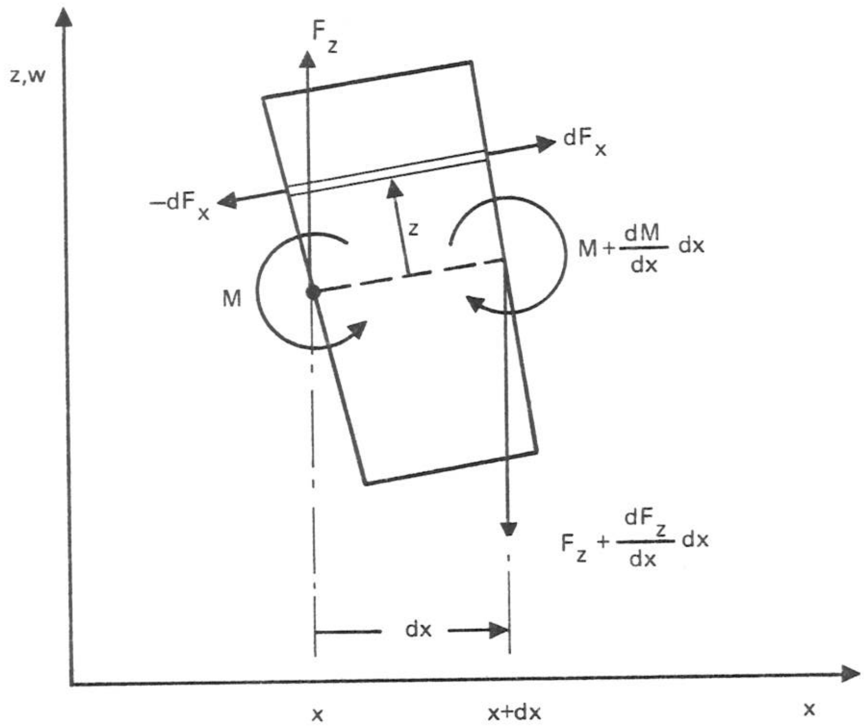

Fig. 5.5. Forces and Moments Associated with Flexure

follows that

$$d\xi = \frac{\partial \xi}{\partial x} dx = z\, d\theta = z \frac{\partial \theta}{\partial x} dx . \tag{5.11}$$

Substituting into Eq. 5.10, the extensional force is

$$dF_x = - Ybz \frac{\partial \theta}{\partial x} dx . \tag{5.12}$$

Although the net extensional force is zero, this force causes a rotational moment about an axis in the neutral plane perpendicular to the plane of the motion, given by

$$M = \int z\, dF_x = \int z \frac{\partial F_x}{\partial z} dz . \tag{5.13}$$

Using Eq. 5.12 for the extensional force,

$$M = - \int Ybz^2 \frac{\partial \theta}{\partial x} dz = - Y \frac{\partial \theta}{\partial x} \int bz^2\, dz . \tag{5.14}$$

The integral in this expression is the *moment of inertia*, I, of the beam cross-sectional area relative to the neutral plane. It is usual to express it as the product of the *cross-sectional area*, S, and the square of the *radius of gyration*, κ. Thus,

$$M = -YI \frac{\partial \theta}{\partial x} = -YS\kappa^2 \frac{\partial \theta}{\partial x} . \tag{5.15}$$

The product of Young's modulus and the moment of inertia is called the *bending stiffness* or *rigidity*, B.

Transverse Acceleration

As shown in Fig. 5.5, the beam element experiences a net perpendicular force, dF_z, which force is related to the transverse acceleration of the element by Newton's second law. Thus,

$$\mu \frac{\partial^2 w}{\partial t^2} dx = -dF_z = -\frac{\partial F_z}{\partial x} dx , \tag{5.16}$$

where μ is the total mass involved in the motion per unit length of the beam, including any entrained mass, m_e, of the fluid as well as the mass of the structure. Defining ϵ as the ratio of the entrained mass to that of the structure,

$$\mu = (1 + \epsilon) \int_S \rho_s \, dS = (1 + \epsilon)\bar{\rho}_s S \tag{5.17}$$

and

$$\frac{\partial F_z}{\partial x} = -\mu \ddot{w} = -(1 + \epsilon)\bar{\rho}_s S \ddot{w} , \tag{5.18}$$

where each dot represents a differentiation with respect to time.

Rotational Acceleration

The perpendicular force, F_z, and the net moment on the element combine to create a rotational torque on the element about an axis through its center of gravity (c.g.) perpendicular to the plane of the motion. This torque causes rotational acceleration of the element. From the rotational form of Newton's second law,

$$I'\ddot{\theta} \, dx = -\left(F_z + \frac{\partial M}{\partial x}\right) dx , \tag{5.19}$$

where I' is the *mass-moment of rotatory inertia* about the c.g. and is given by

$$I' \equiv \int \rho_s b z'^2 \, dz' = \bar{\rho}_s S \kappa'^2 . \tag{5.20}$$

In this expression, z' is measured from the c.g. If the section is symmetric and homogeneous, I' will equal $\rho_s I$. We will find it useful to represent the ratio of I' to the product $\bar{\rho}_s I$ as a non-dimensional *coefficient of relative rotatory inertia*, α', as

$$\alpha' \equiv \frac{I'}{\bar{\rho}_s I} \; . \tag{5.21}$$

Equation 5.19 can now be written

$$\alpha' \bar{\rho}_s I \ddot{\theta} = - F_z - \frac{\partial M}{\partial x} \; . \tag{5.22}$$

Considerations of Shear

Equations 5.15, 5.18 and 5.22 are three independent equations relating the transverse force, F_z, moment, M, section angle, θ, and section displacement, w. A fourth equation is needed in order to solve for w or θ alone. Early investigators, including Rayleigh, assumed that any shear distortion of the element would be negligible and that the slope of the neutral plane would equal the angle of rotation, θ. Timoshenko (1921) was the first to include shear distortion. He noted that the transverse force causes the slope of the neutral plane to be less than θ by an amount given by the shear strain divided by the shear modulus, i.e.,

$$\frac{\partial w}{\partial x} = \theta - \frac{F_z}{KGS} \; , \tag{5.23}$$

where K is a factor, always less than unity, that takes into account warping of the cross section and the lack of constancy of the shear force over the entire section. K depends on both section shape and Poisson's ratio.

The product KGS has the dimensions of a force. It is useful to follow Plass (1958) and define a non-dimensional *shear parameter*, Γ, by

$$\Gamma \equiv \frac{Y}{KG} = \frac{1}{K}\left(\frac{c_\ell}{c_s}\right)^2 = \frac{2(1 + \sigma)}{K} \; . \tag{5.24}$$

Since K is always less than unity, this parameter always exceeds 2. Equation 5.23 relating the slope of the neutral plane to the transverse force can now be written

$$\frac{\partial w}{\partial x} = \theta - \frac{\Gamma}{YS} F_z \; . \tag{5.25}$$

Differential Equation for Bending

Equations 5.15, 5.18, 5.22 and 5.25 can be treated as a set of four coupled equations and their solution found by finite difference, or other computational methods. They can also be combined to form a single fourth-order differential equation for the displacement, w. In its most general form, the resultant equation is quite complicated and of little practical use. However, it can be shown that a number of the terms are invariably small compared to others and can be eliminated without materially affecting the result.

To derive the differential equation for bending, the second derivative with respect to x of Eq. 5.15 and the first derivatives of Eqs. 5.22 and 5.25 are combined with Eq. 5.18, and certain third-order terms involving spatial derivatives of beam dimensions are eliminated. The result is an equation for flexural displacement of a beam that involves only even-order space and time derivatives,

$$\mu\ddot{w} + \frac{\partial^2}{\partial x^2}\left(YI\frac{\partial^2 w}{\partial x^2}\right) - \left(\Gamma(1+\epsilon) + \alpha'\right)\bar{\rho}_s I\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\alpha'\Gamma\bar{\rho}_s\kappa^2\mu}{Y}\ddddot{w} = f(x,t) \,, \tag{5.26}$$

where $f(x,t)$ is any applied force per unit length. Assuming that the material is homogeneous, and using Eq. 5.1 for c_ℓ, the resultant equation is

$$\ddot{w} + \frac{c_\ell^2\kappa^2}{1+\epsilon}\left(\frac{\partial^4 w}{\partial x^4} + \frac{2}{I}\frac{dI}{dx}\frac{\partial^3 w}{\partial x^3} + \frac{1}{I}\frac{d^2I}{dx^2}\frac{\partial^2 w}{\partial x^2}\right) - \left(\Gamma(1+\epsilon) + \alpha'\right)\frac{\kappa^2}{1+\epsilon}\frac{\partial^2\ddot{w}}{\partial x^2}$$
$$+ \frac{\alpha'\Gamma\kappa^2}{c_\ell^2}\ddddot{w} = \frac{f(x,t)}{\mu} \,. \tag{5.27}$$

This differential equation is more general than the equation originally derived by Timoshenko. It includes effects of non-uniformities and of entrained mass as well as rotatory inertia and shear distortion.

Equation for Uniform Beams

Timoshenko's equation applies to uniform beams and also to non-uniform beams for which spatial derivatives of the moment of inertia are negligible. Retaining entrained mass terms, Eq. 5.27 becomes

$$\ddot{w} + \frac{c_\ell^2\kappa^2}{1+\epsilon}\frac{\partial^4 w}{\partial x^4} - \left(\Gamma(1+\epsilon) + \alpha'\right)\frac{\kappa^2}{1+\epsilon}\frac{\partial^2\ddot{w}}{\partial x^2} + \frac{\alpha'\Gamma\kappa^2}{c_\ell^2}\ddddot{w} = \frac{f(x,t)}{\mu} \,. \tag{5.28}$$

This final equation is the basis for analyses of structural vibrations covering a wide frequency range. Its validity has been corroborated by a number of investigators. Huang (1961) found it to give results in excellent agreement with those of exact elasticity theory. Further, Ripperger and Abramson (1957) confirmed Timoshenko's theory as applied to relatively high frequencies by experiments involving the response of beams to hammer blows. Equations 5.27 and 5.28 may therefore be used with confidence over the entire frequency range of interest in structure-borne sound.

Euler-Bernoulli (E-B) Equation

When dealing with relatively thin beams and/or low frequencies, the first two terms in Eq. 5.28 are dominant and the equation takes a simple form:

$$\ddot{w} + \frac{c_\ell^2 \kappa^2}{1 + \epsilon} \frac{\partial^4 w}{\partial x^4} = \frac{f(x,t)}{\mu} \quad . \tag{5.29}$$

This equation was derived independently by Euler and Bernoulli in the 19th century. In their derivations, they ignored rotatory motion of the beam element and also shear distortion, which amounts to setting both α' and Γ equal to zero. Their equation is often quite useful, and many texts use it exclusively when treating flexural vibrations. In what follows we will find that low-frequency limits of more complete expressions are solutions of the E-B equation.

5.4 Speed of Flexural Waves

Harmonic Solutions of the Timoshenko Equation

The complete Timoshenko beam equation can be used to solve for the displacement, w, of the neutral plane for a given force distribution. Also of interest in acoustics is the effective phase speed for flexural motions as a function of frequency. Equation 5.28 is a fourth-order linear differential equation and not strictly a wave equation. However, only even-order derivatives are involved. As a result, when motion at a single frequency is assumed, an effective wave speed can be found.

We may express the complete solution of Eq. 5.28 as the sum of four terms, of the form

$$\underline{w} = \sum_{i=1}^{4} \underline{A}_i \, e^{i(\omega t - k_i x)} \quad . \tag{5.30}$$

Substituting Eq. 5.30 and its derivatives into Eq. 5.28 yields

$$- \omega^2 \underline{A}_i + c_\ell^2 \frac{\kappa^2}{1 + \epsilon} k_i^4 \underline{A}_i - \left(\Gamma(1 + \epsilon) + \alpha'\right) \frac{\kappa^2}{1 + \epsilon} \omega^2 k_i^2 \underline{A}_i$$
$$+ \frac{\alpha' \Gamma \kappa^2}{c_\ell^2} \omega^4 \underline{A}_i = \frac{f(x,\omega)}{\mu} \quad . \tag{5.31}$$

This equation can be used to solve forced vibration problems, provided only that the effects of any non-uniformities of the beam are negligible. Expressions for wave phase speeds can be obtained by considering the homogeneous equation for free vibrations in the absence of any external forces. Setting the forcing function equal to zero results in a quartic algebraic equation for k_i for which there are four solutions. Two of the solutions are real and two are imaginary. The two real solutions are equal in magnitude but opposite in sign and represent flexural waves traveling in opposite directions. We may represent these values of k_i by $\pm k_f$, since k_f has all the properties of a wave number for propagating sinusoidal components. The two imaginary solutions are represented by $\pm i\gamma$. These terms have exponential form and account for non-sinusoidal distortions that occur at discontinuities in the geometry of the structure, especially at the ends. Thus, the solution of the homogeneous form of Eq. 5.31 may be written

$$\underline{w} = \underline{A} e^{-ik_f x} + \underline{B} e^{ik_f x} + \underline{C} e^{-\gamma x} + \underline{D} e^{\gamma x} \quad . \tag{5.32}$$

Alternatively, in terms of sinusoids and hyperbolic functions,

$$\underline{w} = \underline{a} \sin k_f x + \underline{b} \cos k_f x + \underline{c} \sinh \gamma x + \underline{d} \cosh \gamma x \ . \tag{5.33}$$

We will return to these equations when dealing with resonances of finite structures in the next section.

Since the two real solutions are wave solutions, Eq. 5.31 can be transformed into an equation for the flexural wave phase speed, v_f, which is related to the wave number by

$$v_f(\omega) = \frac{\omega}{k_f} \ . \tag{5.34}$$

The homogeneous form of Eq. 5.31 can be written

$$\left(1 - \alpha'\Gamma(1+\epsilon)\left(\frac{\omega}{\Omega}\right)\right)^2 \left(\frac{v_f}{c_\ell}\right)^4 + \left(\Gamma(1+\epsilon) + \alpha'\right)\left(\frac{\omega}{\Omega}\right)^2 \left(\frac{v_f}{c_\ell}\right)^2 - \left(\frac{\omega}{\Omega}\right)^2 = 0 \ . \tag{5.35}$$

where Ω is a reference angular frequency defined by

$$\Omega \equiv c_\ell^2 \sqrt{\frac{\mu}{YI}} = \frac{c_\ell}{\kappa}\sqrt{1+\epsilon} \ . \tag{5.36}$$

This is a non-dimensional form of the complete Timoshenko beam equation, expressed as a quartic for the relative flexural wave speed as a function of a relative frequency and three dimensionless parameters representing the effects of entrained mass, shear distortion and rotatory inertia.

Applying the quadratic theorem, the complete solution of Eq. 5.35 is

$$\left(\frac{v_f}{c_\ell}\right)^2 = \frac{\sqrt{\left(\Gamma(1+\epsilon) - \alpha'\right)^2\left(\frac{\omega}{\Omega}\right)^4 + 4\left(\frac{\omega}{\Omega}\right)^2} - \left(\Gamma(1+\epsilon) + \alpha'\right)\left(\frac{\omega}{\Omega}\right)^2}{2\left(1 - \alpha'\Gamma(1+\epsilon)\left(\frac{\omega}{\Omega}\right)^2\right)} \ . \tag{5.37}$$

This is a complex function of the parameters, which fortunately can be represented quite accurately in terms of approximate solutions applicable at low, intermediate and high frequencies.

Low-Frequency Approximation

Examining Eqs. 5.35 and 5.37, it is apparent that the terms which involve the shear parameter and the relative rotatory inertia are proportional to the square of the relative frequency. These terms are negligible for relatively low frequencies, for which

$$\left(\frac{v_f}{c_\ell}\right)_\ell \doteq \sqrt{\frac{\omega}{\Omega}} = \sqrt{\frac{\omega\kappa}{c_\ell\sqrt{1+\epsilon}}} = \frac{1}{c_\ell}\sqrt[4]{\frac{\omega^2 YI}{\mu}} \qquad \left(\Gamma\left(\frac{\omega}{\Omega}\right)^2 << 1\right). \tag{5.38}$$

This same result could have been derived directly from the homogeneous form of the E-B equation, Eq. 5.29. We can now interpret Ω to be the value of ω for which the flexural wave speed would equal the longitudinal wave speed, if the E-B equation were valid at all frequencies.

Since the flexural wave speed depends on frequency, low-frequency flexural waves are dispersive. The speed v_f is the phase speed for a monochromatic component. If a wideband pulse is transmitted, it will travel with a group velocity, v_g, equal to twice the phase speed at the median frequency, provided Eq. 5.38 is a valid approximation.

High-Frequency Limit

At the very highest frequencies, flexural waves degenerate into shear waves. Although these waves have little practical importance in structure-borne sound, the expression for their phase speed is useful as a limit. In the high-frequency limit, the entrained mass approaches zero, and

$$\left(\frac{v_f}{c_\ell}\right)_h^2 \doteq \frac{1}{\Gamma} = K\left(\frac{c_s}{c_\ell}\right)^2 \qquad \left(\Gamma\left(\frac{\omega}{\Omega}\right) >> 1\right). \tag{5.39}$$

Since the effective shear area is always less than the cross-sectional area, the limiting flexural wave speed is always somewhat lower than the shear wave speed. In terms of the low-frequency, E-B result, the high-frequency limit may be expressed by

$$\frac{v_{f_h}}{v_{f_\ell}} \doteq \frac{1}{\sqrt{\Gamma\left(\frac{\omega}{\Omega}\right)}}. \tag{5.40}$$

Intermediate-Frequency Approximation

Between the lowest and highest frequencies, the wave speed curve, as given by its complete solution in Eq. 5.37, makes a smooth transition from the low-frequency values of Eq. 5.38 to its high-frequency limit given by Eq. 5.39. In this intermediate frequency range the entrained mass has a decreasing effect. In addition, Budiansky and Kruszewski (1953) and others have shown that the effects of shear distortion are always at least three times as great as those of rotatory inertia. These facts can be used in deriving approximate relations for the flexural wave speed for intermediate frequencies. Equation 5.37 can be written

$$\left(\frac{v_f}{c_\ell}\right)^2 = \left(\frac{v_f}{c_\ell}\right)_\ell^2 \left[\frac{\sqrt{1+(1-\bar{\alpha})^2\delta^2} - (1+\bar{\alpha})\delta}{1-4\bar{\alpha}\delta^2}\right], \tag{5.41}$$

where

$$\delta \equiv \frac{\Gamma(1+\epsilon)}{2}\left(\frac{\omega}{\Omega}\right) \tag{5.42}$$

and

$$\bar{\alpha} \equiv \frac{\alpha'}{\Gamma(1 + \epsilon)} . \qquad (5.43)$$

The parameter $\bar{\alpha}$ expresses the magnitude of the rotatory inertia correction relative to that for shear. This factor never exceeds 1/3 and is often smaller. In the limit as $\bar{\alpha} \rightarrow 0$, Eq. 5.41 reduces to

$$\frac{v_f}{v_{f\ell}} \doteq \sqrt{\sqrt{1 + \delta^2} - \delta} . \qquad (5.44)$$

Since the effect of rotatory inertia is relatively small, Eq. 5.44 can be used to calculate corrections to the E-B solution for all frequencies. The correction is not significant for values of δ less than about 0.05. For values of $\delta > 3$, Eq. 5.44 gives values in very close agreement with those from the high-frequency expression, Eq. 5.40. Between these limits it may differ from that given by Eqs. 5.37 and 5.41 by as much as 3%, as shown in Fig. 5.6. Since values of other quantities such as entrained mass are often uncertain, Eqs. 5.38 and 5.44 are recommended for calculation of the flexural wave speed throughout the entire frequency range unless very precise values are required, when Eq. 5.41 should be used.

Solid Rectangular Bars

The equations for bending vibrations derived thus far in this chapter are quite general in that they apply to any type of cross section and include entrained mass. Beam structures having solid,

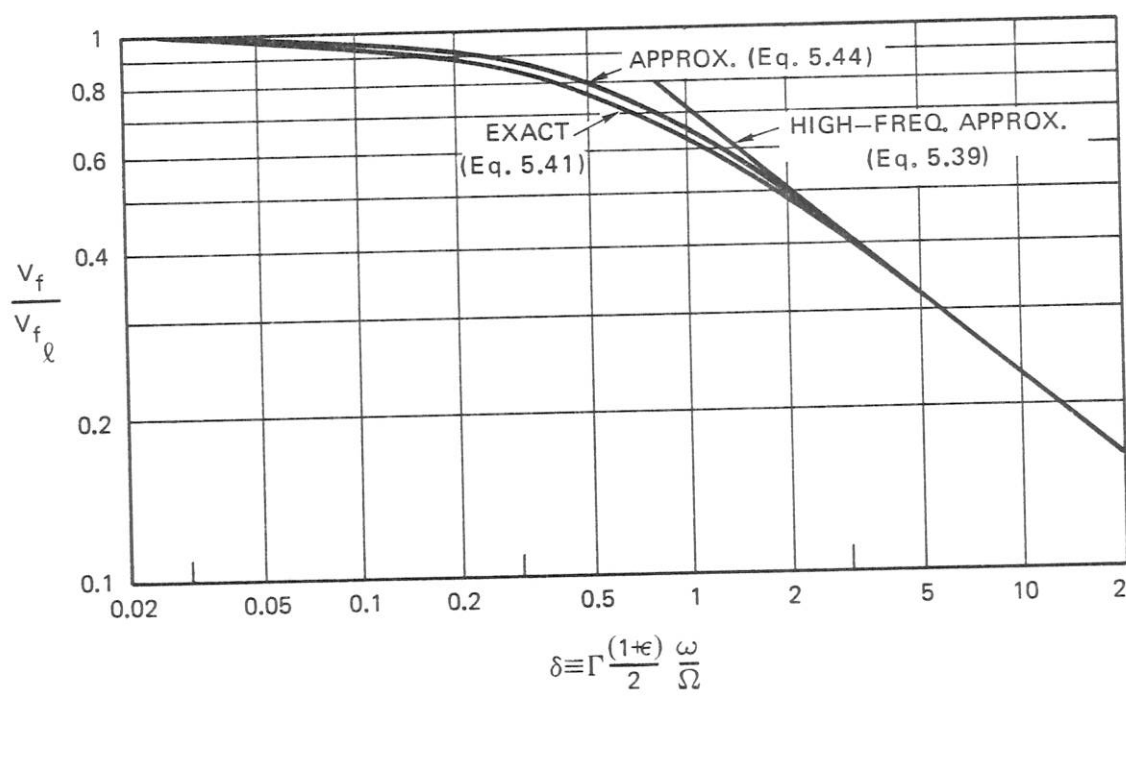

Fig. 5.6. Flexural Wave Speed Relative to Low-Frequency Approximation, for $\bar{\alpha} = 1/3$

rectangular cross sections are frequently encountered in practice. It is instructive to consider the special case of such uniform rectangular beams vibrating *in air*. For solid bodies of rectangular cross section having width b and thickness h,

$$\kappa = \frac{h}{\sqrt{12}} , \tag{5.45}$$

$$\alpha' = 1 , \tag{5.46}$$

$$K \doteq \frac{4 + \sigma}{5} , \tag{5.47}$$

and

$$\Gamma = \frac{2(1 + \sigma)}{K} \doteq 10 \left(\frac{1 + \sigma}{4 + \sigma}\right) \doteq \frac{5}{2}\left(1 + \frac{3}{4}\sigma\right) . \tag{5.48}$$

The other parameters used in the analysis become

$$\Omega = \frac{c_\ell}{\kappa} = \frac{c_\ell}{h}\sqrt{12} \tag{5.49}$$

and

$$\delta = \frac{\Gamma}{2}\frac{\omega}{\Omega} \doteq \frac{5}{8\sqrt{3}}\left(1 + \frac{3}{4}\sigma\right)\frac{\omega h}{c_\ell} . \tag{5.50}$$

The low-frequency approximation of Eq. 5.38 leads to

$$\left(\frac{v_f}{c_\ell}\right)_\ell^2 \doteq \frac{\omega}{\Omega} = \frac{\omega h}{c_\ell\sqrt{12}} , \tag{5.51}$$

and the high-frequency limiting value is

$$\left(\frac{v_f}{c_\ell}\right)_h^2 \doteq \frac{1}{\Gamma} \doteq \frac{4 + \sigma}{10(1 + \sigma)} \doteq \frac{0.4}{1 + \frac{3}{4}\sigma} . \tag{5.52}$$

The low-frequency solution can also be expressed in terms of the flexural wave number, k_f, by dividing both sides of Eq. 5.51 by v_f, giving

$$\frac{v_{f\ell}}{c_\ell} = \frac{\omega h}{v_{f\ell}\sqrt{12}} = \frac{k_f h}{\sqrt{12}} . \tag{5.53}$$

The intermediate-frequency correction given by Eq. 5.44 can be used for solid rectangular bars with δ given by Eq. 5.50. However, for metal bars the more exact solution of Eq. 5.41 has an especially simple form. For most metals, the shear parameter, Eq. 5.48, is close to 3. Taking $\Gamma = 3$ and $\alpha' = 1$, Eq. 5.37 reduces to

$$\left(\frac{v_f}{v_{f\ell}}\right)^2 \doteq \frac{\sqrt{1 + \left(\frac{\omega}{\Omega}\right)^2} - 2\,\frac{\omega}{\Omega}}{1 - 3\left(\frac{\omega}{\Omega}\right)^2} . \qquad (5.54)$$

This result is plotted in Fig. 5.7. Also shown is the high-frequency limit and a simple approximate formula,

$$\frac{v_f}{v_{f\ell}} \doteq \frac{1}{\sqrt{1 + \frac{5}{2}\left(\frac{\omega}{\Omega}\right)}} . \qquad (5.55)$$

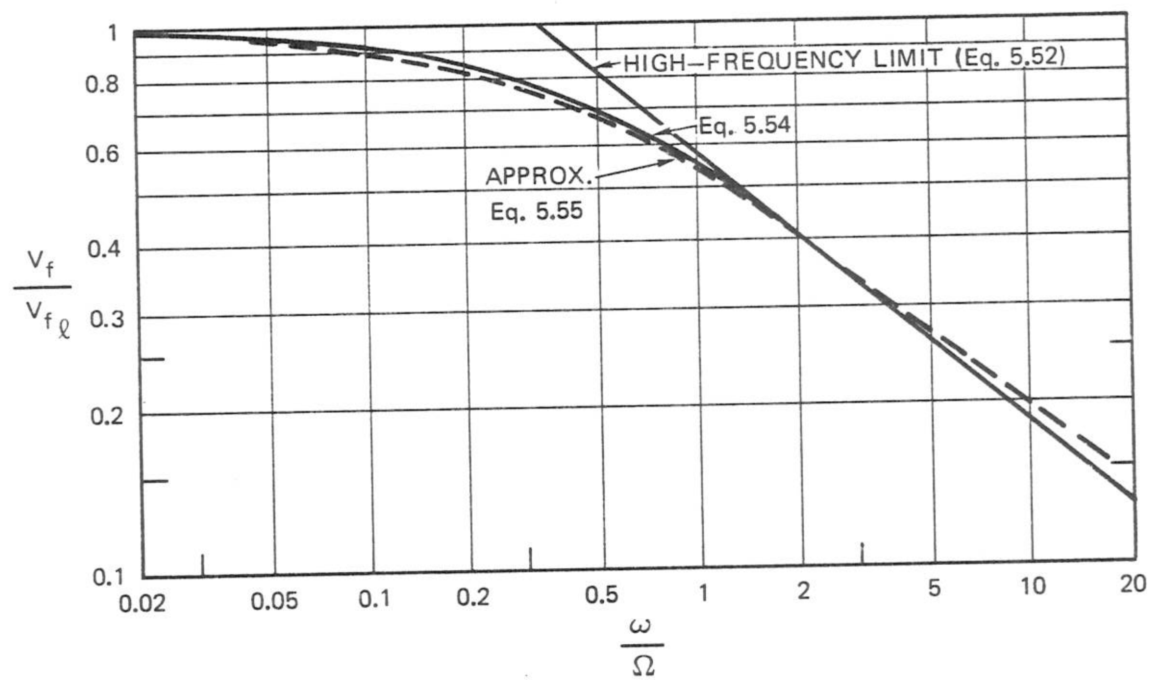

Fig. 5.7. Relative Flexural Wave Speed of Solid Metal Rectangular Bars

Nelson (1971) derived a universal dispersion curve for solid rectangular bars which he expressed in terms of the flexural wave number, k_f. His results for metal bars can be represented by the expression

$$\frac{v_f}{v_{f\ell}} \doteq \frac{2}{1 + \sqrt{1 + \frac{3}{4}(k_f h)^2}} \qquad (5.56)$$

up to $k_f h \doteq 8$, as shown in Fig. 5.8. Above $k_f h = 8$, Nelson's values are in good agreement with

$$\frac{v_{f_h}}{v_{f\ell}} \doteq \frac{2}{k_f h} \; . \qquad (5.57)$$

Most of the expressions for flexural wave speeds given in the present section on rectangular bars can be applied to solid rods having circular cross sections. Since the radius of gyration is

$$\kappa = \frac{D}{\sqrt{8}} \; , \qquad (5.58)$$

it is only necessary to replace h by $\sqrt{(3/2)}D$ wherever it occurs.

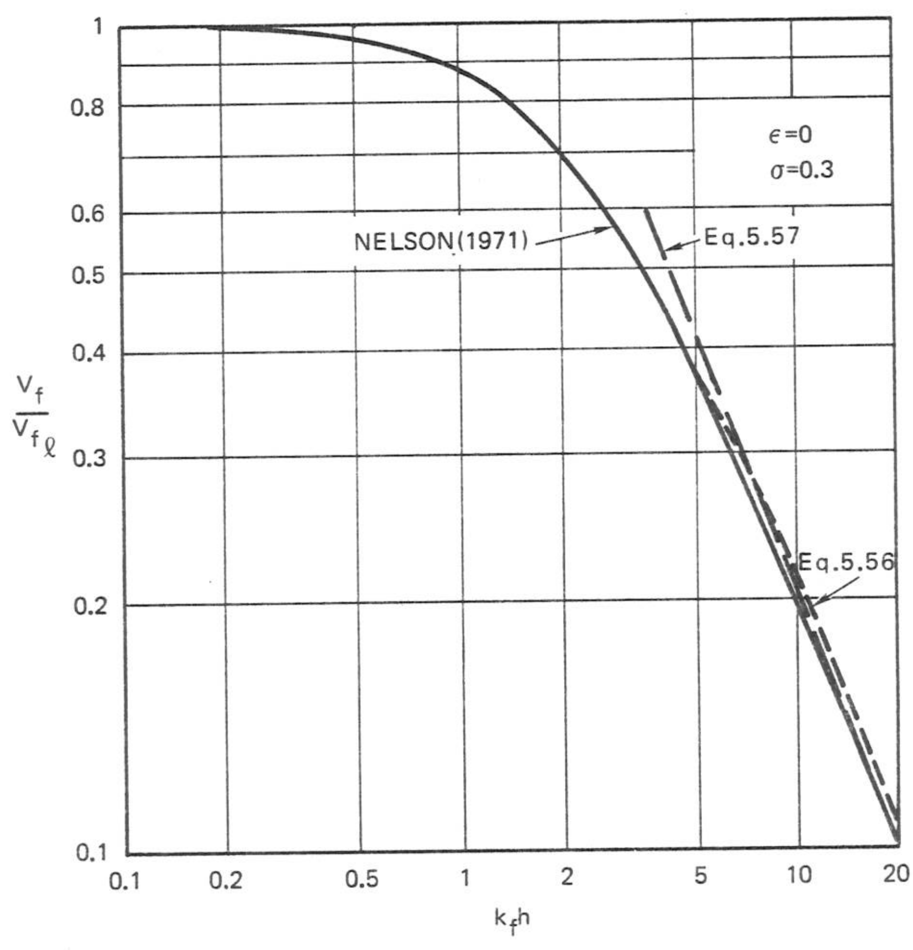

Fig. 5.8. Dispersion Curves for Rectangular Metal Beams

5.5 Flexural Resonances

Just as an organ pipe has resonant frequencies which are related to the ratio of its length to the acoustic wavelength, so finite length beams have resonant frequencies for flexural vibrations. Beam resonant frequencies depend upon method of support, much as pipe resonances depend on whether the pipe end is open or closed. Since flexural resonances of finite structures play an important role in structure-borne sound, considerable attention has been given to this subject in the literature.

Uniform Thin Beams

There are two approaches that can be used to find resonances of uniform beams. The more common one starts from the assumption that the solution of the differential equation involves four terms, as given by Eqs. 5.32 and 5.33. Values of the four coefficients can be found by using either of these two equations, its first three derivatives, and mathematical expressions for the physical conditions at the ends. Resonance frequencies are then determined by assuming that the applied force is zero.

In the second approach, a wave is considered to travel from one end, to reflect from the other end, and to reflect again from the first. Resonance occurs when the wave that has completed a round trip is in phase with a wave that is just starting out. In this approach, the beam equation is used to find the flexural wave speed, from which the time of travel can be computed. The boundary conditions at the ends merely act to impose phase shifts. Both methods will be developed. The first is more usual and the second has the advantage that it can be readily applied to many non-uniform beams.

In the frequency regime for which the E-B equation, Eq. 5.29, is valid, the complete solution of the homogeneous equation is as given by Eqs. 5.32 and 5.33 except that $\gamma = k_f$. From Eq. 5.38,

$$k_f \doteq \frac{\omega}{v_{f\ell}} = \frac{1}{c_\ell}\sqrt{\omega\Omega} = \sqrt{\frac{\omega}{\kappa c_\ell}\sqrt{1+\epsilon}} = \sqrt[4]{\frac{\mu\omega^2}{YI}} \,. \tag{5.59}$$

For an infinite beam, all values of k_f are valid solutions of the homogeneous equation. However, for a finite beam only certain values can occur and these are dependent on the end conditions.

The mathematical end conditions depend on the physical nature of the end attachments. There are three basic end conditions generally considered:

a) free,

b) clamped, and

c) simply supported.

For a *free end*, no requirement is imposed on w or on its first derivative. However, both the moment and the force must be zero. From the derivation of the bending equation in Section 5.3, it can readily be shown that these conditions require that the second and third derivatives of w be zero. For a *clamped end*, both w and its first derivative must be zero. For a *simply-supported end*, both w and the moment must be zero, from which the second derivative of w must also be zero.

With these end conditions and simultaneous solution of the resultant equations, it can readily be shown that, for free and/or clamped end conditions, the condition for resonance of a beam of length L is given by

$$\cos k_f L \cosh k_f L = \pm 1 \ , \tag{5.60}$$

where the plus sign applies to bars that are free or clamped at both ends, and the minus sign applies to a beam that is free at one end and clamped at the other. If both ends are simply supported, the corresponding resonance condition is

$$\sin k_f L = 0 \ . \tag{5.61}$$

The frequencies that satisfy Eqs. 5.60 or 5.61 are the resonance frequencies.

Except for the lowest frequency resonances, the beam length is large compared to a flexural wave length, i.e., $k_f L >> 1$. The hyperbolic cosine in Eq. 5.60 is therefore large compared to unity, and the resonance condition is simply that $\cos k_f L$ be zero. Thus, for free and clamped ends, the condition for resonance is

$$k_{f_m} L \doteq (2m - 1) \frac{\pi}{2} \ , \tag{5.62}$$

where the index, m, indicates the number of nodes that occur along the length of the beam. Solving for the resonance frequency,

$$f_m = \frac{\omega_m}{2\pi} = k_{f_m}^2 L^2 \frac{\kappa c_\ell}{2\pi L^2 \sqrt{1 + \epsilon}} \doteq (2m - 1)^2 \frac{\pi}{8} \frac{\kappa c_\ell}{L^2 \sqrt{1 + \epsilon}} \ . \tag{5.63}$$

It follows that, in the low-frequency regime, the frequency separation between resonances increases linearly with frequency.

For a beam free at one end but clamped at the other, m can be any integer starting with 1. However, the case $m = 1$ has to be ruled out for a beam free at both ends (free-free) because it would imply a rigid-body translation or rotation of the beam. Since no external forces or moments are allowable in a resonance condition, this case is not admissible. The lowest natural frequency of a free-free beam is therefore the two-noded one, for which $m = 2$. Figure 5.9 illustrates some of

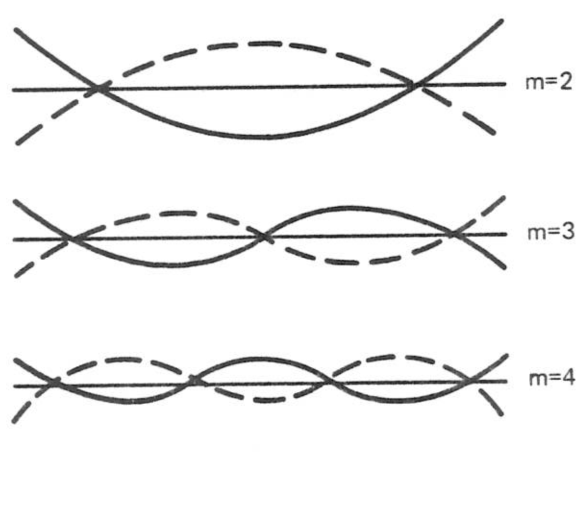

Fig. 5.9. Low-Order Resonances of a Free-Free Beam

the lower order resonances of a free-free beam. Since a clamped end is a node, the lowest order resonance of a clamped-clamped beam is also the two-noded one. The resonances of clamped-clamped beams are therefore also given by Eq. 5.63 with $m \geqslant 2$.

The resonance frequencies for free and clamped thin beams given by Eq. 5.63 are quite accurate for $m \geqslant 3$. However, the values for $m = 1$ and $m = 2$ are somewhat in error. The reason for this is that at the lowest resonances the hyperbolic cosine is not sufficiently large and the cosine, while small, can not be set equal to zero. Equation 5.63 can be made to give correct resonance frequencies if m is adjusted slightly. The adjusted values for m are given in Table 5.1.

Table 5.1.
Resonance Conditions for Thin Beams Using Equation 5.63

No. of Nodes	Clamped-Free	Free-Free Clamped-Clamped
1	m = 1.0968	———
2	m = 1.994	m = 2.0056
3	m = 3.000	m = 3.000

The resonance condition for a simply-supported beam is satisfied by

$$k_{f_m} L = (m - 1)\pi \quad (m \geqslant 2) \ , \tag{5.64}$$

from which the resonance frequencies are

$$f_m = \frac{\omega_m}{2\pi} \doteq (m - 1)^2 \frac{\pi}{2L^2} \frac{\kappa c_\ell}{\sqrt{1 + \epsilon}} \ , \tag{5.65}$$

and no corrections are required, even for the fundamental.

The upper frequency limit for applicability of thin-beam resonance conditions, $\delta < 0.03$, can be translated into a limitation on the order of the resonance. Thus, thin-beam approximations are valid provided

$$m < \frac{0.08L}{\kappa\sqrt{\Gamma}} \ . \tag{5.66}$$

Higher order resonances require use of the complete Timoshenko equation.

Correction for Shear and Rotatory Inertia

The thin-beam approximations for resonance frequencies are only valid as long as the flexural wave speed is given closely by the E-B equation. At higher frequencies, shear and rotatory inertia influence the result in two ways: first, the phase speed is reduced relative to the E-B value; second, the effects of the ends are increased, introducing a phase shift.

The effect of the flexural wave speed on the wave number is readily calculated from the results of the previous section. Thus, from Eq. 5.44,

$$k_f = \frac{\omega}{v_f} = k_{f_o}\frac{v_{f\ell}}{v_f} \doteq \frac{k_{f_o}}{\sqrt{\sqrt{1+\delta^2}-\delta}} = k_{f_o}\sqrt{\sqrt{1+\delta^2}+\delta}\ , \qquad (5.67)$$

where k_{f_o} is the low-frequency approximation, as given by Eq. 5.59. The parameter γ which controls the space-rate-of-decay of the influence of the ends is given by

$$\gamma = k_{f_o}\frac{v_f}{v_{f\ell}} = k_{f_o}\sqrt{\sqrt{1+\delta^2}-\delta} = k_f\left(\sqrt{1+\delta^2}-\delta\right)\ . \qquad (5.68)$$

It is to be noted that the low-frequency wave number, k_{f_o}, equals the geometric mean of γ and k_f.

Leibowitz and Kennard (1964) have shown that when end conditions are applied to the full Timoshenko solution, Eq. 5.33, and its derivatives, the resonance condition for beams with free and/or clamped ends becomes

$$\pm 1 = \cos k_fL \cosh \gamma L + \left(\frac{k_f^2-\gamma^2}{2\gamma k_f}\right)\sin k_fL \sinh \gamma L\ . \qquad (5.69)$$

Substituting for γ from Eq. 5.68, one finds

$$\pm 1 = \cos k_fL \cosh \gamma L + \delta \sin k_fL \sinh \gamma L\ , \qquad (5.70)$$

from which it can be shown that, for $m \geqslant 2$,

$$k_{f_m}L \doteq (2m-1)\frac{\pi}{2} + \tan^{-1}\delta\ . \qquad (5.71)$$

Equation 5.71 is identical to Eq. 5.62 except for the phase shift. The effect of this phase shift is to raise the resonance frequency from its low-frequency value. However, the effects of shear and rotatory inertia on the wave number itself, as expressed by Eq. 5.67, are much greater. The net result is that the resonances in the intermediate-frequency region are lower in frequency than they would be if the thin-beam solution were applicable. Since δ is itself a function of the frequency, solutions for resonance frequencies require an iterative process. A formula which fits the results very well is

$$\frac{f_m}{f_{m\ell}} = \frac{\omega_m}{\omega_{m\ell}} \doteq \frac{2}{1+\sqrt{1+6\left(1-\frac{4}{(2m-1)\pi}\right)\delta_{m\ell}}}\ , \qquad (5.72)$$

where δ_ℓ is calculated in terms of the low-frequency estimate of resonance by

$$\delta_{m\ell} = \frac{\Gamma(1+\epsilon)}{2}\frac{\omega_{m\ell}}{\Omega} = \frac{\Gamma\sqrt{1+\epsilon}}{2}\frac{\omega_{m\ell}\kappa}{c_\ell} = \frac{\Gamma}{2}\left((2m-1)\frac{\pi}{2}\frac{\kappa}{L}\right)^2\ . \qquad (5.73)$$

Substituting this expression into Eq. 5.72 leads to a formula for the correction factor in terms of the order of the resonance and geometric factors,

$$\frac{f_m}{f_{m\ell}} = \frac{\omega_m}{\omega_{m\ell}} \doteq \frac{2}{1 + \sqrt{1 + 3\Gamma(m-1)^2\pi^2 \dfrac{\kappa^2}{L^2}}} \, . \tag{5.74}$$

This same expression also applies to resonances of simply-supported beams. It is graphed in Fig. 5.10.

In the limit at high frequencies,

$$\omega_{m_h} = k_{f_m} v_{f_h} \doteq \frac{(m-1)\pi}{L} \frac{c_\ell}{\sqrt{\Gamma}} \, , \tag{5.75}$$

and

$$\frac{\omega_{m_h}}{\omega_{m\ell}} \doteq \frac{1}{\pi(m-1)\dfrac{\kappa}{L}\sqrt{\Gamma}} = \frac{L}{(m-1)\pi}\sqrt{\frac{KGS}{YI}} \, . \tag{5.76}$$

This relation is also plotted in Fig. 5.10. Equation 5.74 can be rewritten in terms of the ratio of the two limiting values of resonance frequency as

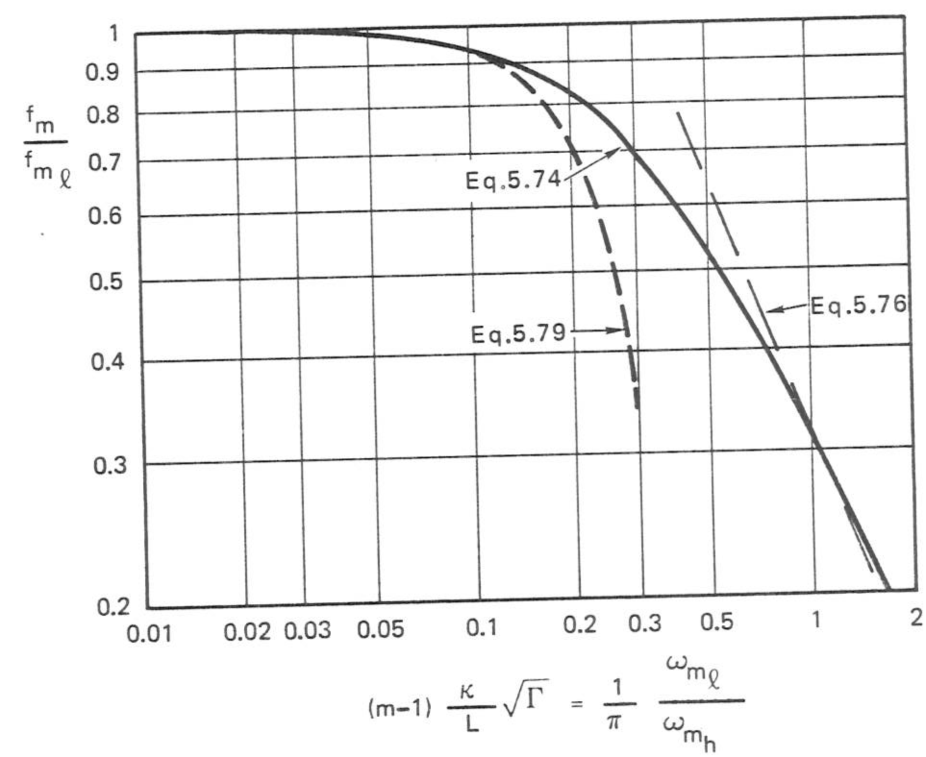

Fig. 5.10. Correction Factor for Beam Resonance Frequencies

$$\frac{f_m}{f_{m\ell}} = \frac{\omega_m}{\omega_{m\ell}} \doteq \frac{2}{1 + \sqrt{1 + 3\left(\omega_{m\ell}/\omega_{m_h}\right)^2}} \;. \tag{5.77}$$

Thus, the procedure to be used in calculating flexural resonances is as follows: first, calculate the E-B value from Eq. 5.63 or 5.65; next, calculate the high-frequency limit for the same order from Eq. 5.75, or its ratio to the low-frequency value from Eq. 5.76; and, finally, correct the E-B value from Fig. 5.10 or Eq. 5.77.

For solid rectangular bars having $\sigma \doteq 0.3$, Eq. 5.74 reduces to

$$\frac{f_m}{f_{m\ell}} = \frac{\omega_m}{\omega_{m\ell}} \doteq \frac{2}{1 + \sqrt{1 + 3(m-1)^2\left(\frac{\pi}{2}\right)^2\left(\frac{h}{L}\right)^2}} \;. \tag{5.78}$$

This result differs significantly from the formula originally derived by Timoshenko (1921) and published in a number of texts. The Timoshenko formula,

$$\frac{f_m}{f_{m\ell}} = \frac{\omega_m}{\omega_{m\ell}} \doteq 1 - \frac{3}{4} m^2 \left(\frac{\pi}{2}\right)^2 \left(\frac{h}{L}\right)^2 , \tag{5.79}$$

agrees with Eq. 5.78 only up to a 10% correction. For higher frequencies it overestimates the correction by increasing amounts, as shown in Fig. 5.10.

Wave Approach

In the approach to resonances used thus far, boundary conditions at the ends were used to evaluate the four coefficients of the complete solution for uniform beams and thus to determine resonances. The wave approach to resonance calculation allows a single unified solution independent of the frequency regime, and it has the important advantage that it can readily be applied to non-uniform beams.

In the wave approach, emphasis is placed on the two wave terms of Eq. 5.32 and resonance is found in the same manner as for an organ pipe or for standing waves on a string. The boundary conditions at each end are assumed to act independently and to introduce phase shifts between incident and reflected waves. The condition for resonance is that a wave that has made a round trip and is starting out again be exactly in phase with a wave being generated. This is the well-known condition for standing waves in any kind of linear resonator. Stated differently, the condition for resonance is that the change in phase of the motion at one location in the time that the wave makes a complete round trip be equal to the phase shifts suffered by the traveling wave upon reflection from the two ends. The phase shift at a fixed point during the time of travel, T, is

$$\Phi_o = \omega T \;. \tag{5.80}$$

Resonance occurs when

$$\Phi_o - (\Delta\phi_1 + \Delta\phi_2) = 2\pi m \;. \tag{5.81}$$

The time of travel, T, is obtained by integrating the reciprocal of the velocity,

$$T = 2 \int_0^L \frac{1}{v_f} dx \; . \qquad (5.82)$$

Combining Eqs. 5.80, 5.81 and 5.82, the general resonance condition for any type of plane wave motion may be written

$$\frac{1}{2} \Phi_o = \omega_m \int_0^L \frac{1}{v_f} dx = \int_0^L k_{f_m} dx = m\pi + \overline{\Delta\phi} \; , \qquad (5.83)$$

where $\overline{\Delta\phi}$ is the average of the phase shifts at the two ends. The integer m equals the number of nodes of the vibration along the structure.

The problem of calculating resonances using Eq. 5.83 reduces to the two separable tasks of determining the wave speed, or wave number, and finding the average phase shifts. The wave speed or wave number can be calculated using any of the previously developed approximations. Except at the very lowest resonances, $m = 1$ to $m = 3$, the two ends may be considered to act independently. The exponential term is zero for a simply-supported end, producing a $-180°$ phase shift at all frequencies.

The situation is not so simple for free and clamped end conditions. The reduction of γ relative to k_f, which occurs for thick beams, implies a change in the influence of the exponential terms. Not only does the influence extend further from each end, but also the relative amplitude increases. As an example, consider a clamped end at $x = 0$. The expressions for the displacement and the slope are

$$\underline{w}(0) = \underline{A} + \underline{B} + \underline{C} \; . \qquad (5.84)$$

and

$$\frac{\partial \underline{w}}{\partial x}(0) = - ik_f L(\underline{A} - \underline{B}) - \gamma L \underline{C} \; . \qquad (5.85)$$

Setting both equal to zero,

$$\underline{A} = \underline{B} \frac{1 - i\left(\frac{\gamma}{k_f}\right)}{1 + i\left(\frac{\gamma}{k_f}\right)} = \underline{B} \frac{1 - \left(\frac{\gamma}{k_f}\right)^2 - 2i\left(\frac{\gamma}{k_f}\right)}{1 + \left(\frac{\gamma}{k_f}\right)^2} \; . \qquad (5.86)$$

The phase angle introduced by the reflection equals the angle whose tangent is the imaginary part of the ratio of A to B divided by its real part:

$$\Delta\phi = \tan^{-1} \frac{-2\left(\dfrac{\gamma}{k_f}\right)}{1 - \left(\dfrac{\gamma}{k_f}\right)^2} = -\tan^{-1}\left[\frac{2\left(\sqrt{1+\delta^2}-\delta\right)}{1-\left(\sqrt{1+\delta^2}-\delta\right)^2}\right] = -\tan^{-1}\frac{1}{\delta} \ . \tag{5.87}$$

Since δ, defined by Eq. 5.42, is a function of frequency, the phase shift varies with frequency, from $-\pi/2$ at low frequencies to zero at the highest frequencies. The phase shift at a free end is also given by Eq. 5.87.

For uniform beams, application of Eq. 5.83 with the appropriate phase shifts leads to resonance conditions identical to those derived by the more common method, as expressed by Eq. 5.71 for free and/or clamped ends and by Eq. 5.64 for simply-supported ones.

5.6 Non-Uniform Beams

Turbine blades and ship hulls are examples of beam structures whose resonance frequencies are affected by non-uniformities of their cross sections. There are two ways in which these non-uniformities act to alter flexural wave speeds and hence resonance frequencies. Variations of the radius of gyration along the beam result in different values of the flexural wave speed at each section, as computed by the formulas of Section 5.4. In addition, changes of the moment of inertia, I, add terms to the basic differential equation. Thus, the correct differential equation for non-uniform beams, Eq. 5.27, includes two terms involving derivatives of I that are not considered in Timoshenko's equation, Eq. 5.28.

A number of approaches have been used to find resonances of non-uniform beams. Some beams vary smoothly and solutions can be found by analytic methods. Other cases are better treated by dividing the structure into a number of finite elements and solving the resultant network matrix. The author's wave approach to finding resonances described in the previous section can also be applied to non-uniform beams. It is relatively simple to use and can often replace other methods.

Finite-Element Methods

In the finite-element approach, the structure is divided into as few as 4 to as many as 400 discrete elements, and the basic bending relations are expressed as a set of four equations for each element. The resultant matrix of simultaneous equations is then solved by an analog or digital computer. Resonance frequencies are those frequencies for which the resultant solution agrees with the end conditions.

When dealing with a finite section of length Δx, Eqs. 5.15, 5.18, 5.22 and 5.25, which relate forces and moments to displacements and angles, may be written in the form

$$\Delta\theta = -\left(\frac{\Delta x}{YI}\right)M \ , \tag{5.88}$$

$$\Delta F_z = (\mu\Delta x)\omega^2 w \ , \tag{5.89}$$

$$(I'\Delta x)\omega^2\theta = F_z\Delta x + \Delta M \,, \tag{5.90}$$

and

$$\Delta w = \theta\Delta x - \left(\frac{\Delta x}{KGS}\right) F_z \,. \tag{5.91}$$

The computation is started by assuming a frequency and assigning arbitrary values to the non-zero quantities at one end and zero to the others. The four equations are then used to calculate the changes over length Δx. The procedure is followed until the other end is reached. It is then repeated for other values of frequency. Those values of frequency which give the proper values at the second end are resonance frequencies.

Prior to the development of high-speed digital computers, McCann and MacNeal (1950) and Trent (1950) independently developed electrical analogy methods for solving vibrating-beam problems. In these methods, the expressions in parentheses in Eqs. 5.88-5.91 are represented by electric circuit elements, usually inductances and capacitances. Two nets are used, coupled by transformers. One net deals with forces and involves mass and shear; the other has elements for bending rigidity and rotatory inertia related to Eqs. 5.88 and 5.90. Figure 5.11 shows such a coupled circuit for a beam element. With the electric analog, resonances are indicated by peaks of the voltages that occur as frequency is varied. Damping can be taken into account by adding resistances. The problem with this method is to pick a scaling factor such that the circuit elements will have reasonable values.

Other finite-element methods assume a deflection curve and use variational techniques to determine modal frequencies.

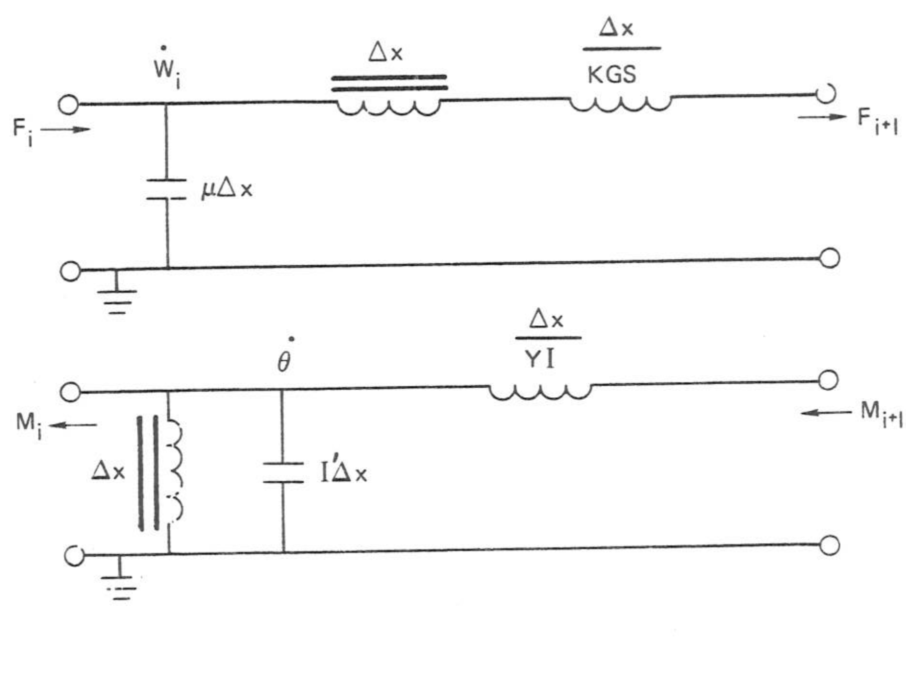

Fig. 5.11. Electric Circuit Analogy for Section of a Vibrating Beam

Wave Method

The wave approach described in the previous section can be extended to non-uniform beams provided that the terms in Eq. 5.27 that involve derivatives of the moment of inertia are small. Substituting Eq. 5.44 for v_f and Eq. 5.38 for $v_{f\ell}$ into Eq. 5.83, the expression for resonance frequencies of a moderately non-uniform beam is

$$\int_0^L \sqrt{\frac{\sqrt{1+\epsilon}}{\kappa c_\ell}\left(\sqrt{1+\delta^2}+\delta\right)}\,dx$$

$$= \int_0^L \sqrt[4]{\frac{\mu}{YI}\left(\sqrt{1+\delta^2}+\delta\right)^2}\,dx = \frac{m\pi + \overline{\Delta\phi}}{\sqrt{\omega_m}} . \tag{5.92}$$

Since the parameter δ, defined by Eq. 5.42, is a function of frequency as well as of position, solution of Eq. 5.92 requires an iterative procedure.

There are several practical ways of solving Eq. 5.92. In one method, the integral is evaluated at a number of frequencies spanning the range of interest and the result plotted as a function of ω. The right-hand side is also plotted as a function of ω for a number of mode numbers. Each intersection represents a resonance frequency. In a second method, the effect of non-uniformity is separated from that of shear. The low-frequency approximation for the resonance frequency is found from

$$\omega_{m\ell} = \left(\frac{m\pi + \overline{\Delta\phi_\ell}}{\int_0^L \sqrt{\frac{\sqrt{1+\epsilon}}{\kappa c_\ell}}\,dx}\right)^2 , \tag{5.93}$$

and the high-frequency approximation from

$$\omega_{m_h} = \frac{c_\ell\left(m\pi + \overline{\Delta\phi_h}\right)}{\int_0^L \sqrt{\Gamma}\,dx} = \frac{c_s\left(m\pi + \overline{\Delta\phi_h}\right)}{\int_0^L \frac{1}{\sqrt{K}}\,dx} . \tag{5.94}$$

Either Eq. 5.77 or Fig. 5.10 is then used to estimate the correction for shear in terms of the ratio of the two limiting values.

An advantage of the wave method relative to most finite-element methods is that, when carrying out the integrations, the beam can be divided into natural elements rather than into even parts. Another advantage is that the effects of non-uniformity and of shear are clearly distinguishable. The major disadvantage is that it is not valid when derivatives of the moment of inertia are important or when end conditions at one end affect the other. Both of these problems occur only at the lowest frequencies, i.e., for the lowest-order modes.

Tapered Cantilever Beams

The limitations of the wave approach can best be understood by examining the extreme case of a doubly-tapered cantilever beam. Such a beam, having rectangular cross section, is shown in Fig. 5.12. It is usual to express each resonance of a tapered beam in terms of that of a uniform

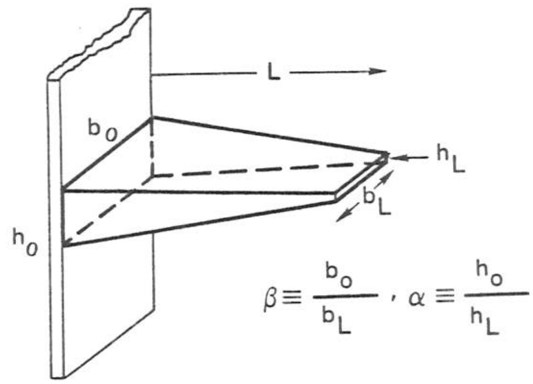

Fig. 5.12. Doubly-Tapered Cantilever Beam

beam having the same dimensions as its base. Using the wave approach and ignoring any effects of shear distortion, this ratio is

$$\left(\frac{\omega_m}{\omega_m(0)}\right)_\ell = \frac{L^2/\kappa_o}{\left(\frac{1}{L}\int_0^L \frac{dx}{\sqrt{\kappa}}\right)^2} = \frac{1}{\left(\int_0^L \sqrt{\frac{h_o}{h}}\, d\left(\frac{x}{L}\right)\right)^2} . \tag{5.95}$$

Expressing the thickness, h, by

$$\frac{h}{h_o} = \frac{1}{\alpha}\left(1 + (\alpha - 1)\left(1 - \frac{x}{L}\right)\right) , \tag{5.96}$$

it follows that

$$\left(\frac{\omega_m}{\omega_m(0)}\right)_\ell = \left(\frac{\alpha - 1}{2\alpha - 2\sqrt{\alpha}}\right)^2 . \tag{5.97}$$

This result is independent of both the resonance order and taper of the width. Also, it predicts that resonance frequencies of tapered beams should always be lower than those of a uniform beam having dimensions of the base throughout.

Martin (1956) measured the resonances of a number of tapered beams. He found that the fundamental increases with increasing taper, while all of the harmonics decrease. Mabie and Rogers (1964, 1972 and 1974) calculated the resonance frequencies of doubly-tapered cantilever beams, starting with the differential equation

$$\mu \ddot{w} + YI\left(\frac{\partial^4 w}{\partial x^4} + \frac{2}{I}\frac{dI}{dx}\frac{\partial^3 w}{\partial x^3} + \frac{1}{I}\frac{d^2 I}{dx^2}\frac{\partial^2 w}{\partial x^2}\right) = 0 \ , \qquad (5.98)$$

which accounts for changes of the cross section but not for shear distortion. Their results are summarized in Fig. 5.13. It is clear from these results that derivatives of the moment of inertia are important for the first four or five resonances, and that the wave approach solution given by Eq. 5.97 is accurate for higher order resonances.

Since doubly-tapered cantilever beams represent an extreme, it appears that derivatives of the moment of inertia can be neglected when dealing with moderately non-uniform beams, especially when calculating resonances involving five or more nodes. This matter is discussed further in Section 5.11 on the flexural resonances of ship hulls.

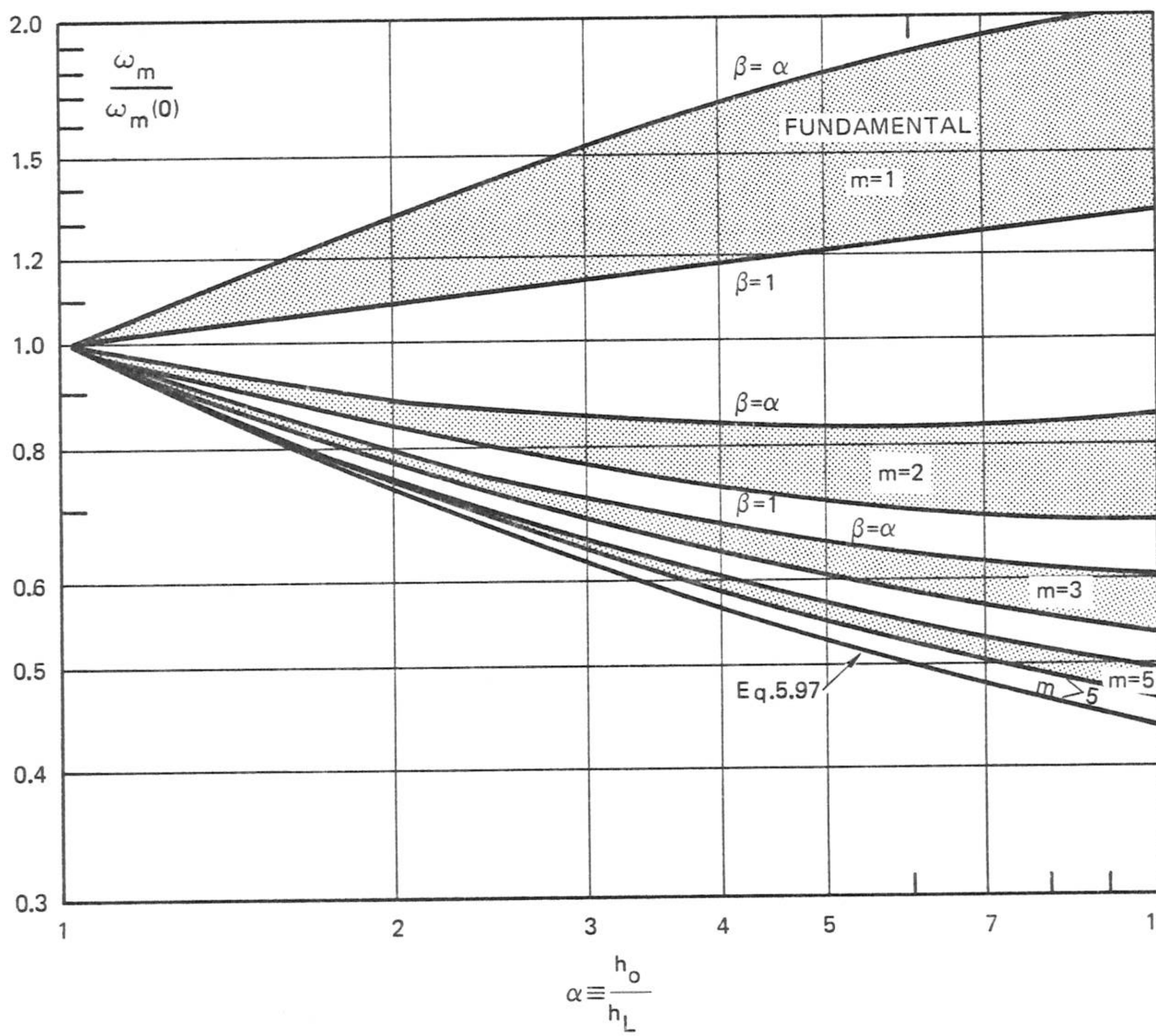

Fig. 5.13. Effect of Taper on Resonance Frequencies of Cantilever Beams, as computed by Mabie and Rogers (1972, 1974)

5.7 Forced Vibrations of Non-Resonant Structures

Mechanical Impedances

As discussed in the introduction of this chapter, beam-like structures often act as transmitters of vibrations from sources to radiating surfaces. The response of beams to exciting forces and their properties as transmitters of vibrations are commonly described in terms of input and transfer impedances and/or admittances, or mobilities. Since the differential equations describing bending are linear, vibratory responses are proportional to exciting forces. *Mechanical impedances* are defined as ratios of exciting forces to resultant structural vibratory velocities. Thus,

$$\underline{Z}(\omega) \equiv \frac{\underline{F}(\omega)}{\underline{\dot{w}}(\omega)} \quad . \tag{5.99}$$

If the velocity is measured at the point of application of the force, the resultant is called the *input impedance*. If the velocity is measured at a different location, then a *transfer impedance* is determined. Since force and velocity are generally not in phase, impedance is a complex quantity consisting of a real, or resistive, component and an imaginary, or reactive, term.

The resistive component of the impedance controls the flow of power in the system. If the source is a vibratory velocity, then the power transferred to the structure is given by

$$W = \underline{F}^* \cdot \underline{\dot{w}}_i = RP(\underline{Z})\overline{\dot{w}_i^2} = R\overline{\dot{w}_i^2} \quad . \tag{5.100}$$

If the source is in the nature of a force, then the power transferred is

$$W = \underline{w}^* \cdot \underline{F}_i = RP\left(\frac{1}{\underline{Z}}\right)\overline{F_i^2} = \frac{R}{|Z|^2}\,\overline{F_i^2} \quad . \tag{5.101}$$

The reciprocal of impedance is called either *admittance* or *mobility*. The power absorbed or transferred when the source is a force is therefore proportional to the real part of the admittance, called *conductance* in analogy to electric circuits.

While almost all structures have resonances, their impedances as non-resonant structures have special significance. Thus, when dealing with excitation of one end of a finite beam, the solution involves its impedance as a semi-infinite beam, and excitation far away from a beam end is related to that for an infinite beam. The two derivations are closely related. We will first consider semi-infinite beams.

Semi-Infinite Beams

A *semi-infinite beam* is defined as a beam that is not only long relative to a flexural wave length but also sufficiently long that any wave reflected from the far end would be damped out and therefore negligible at the near end. Thus, in the vicinity of the near end, only two terms are required to describe the motion. Equation 5.32 can be written

$$\underline{w}(x) \doteq \underline{A}\, e^{-ik_f x} + \underline{C}\, e^{-\gamma x} \quad . \tag{5.102}$$

Since the beam must be free to respond to a force at its end, the moment there must be zero. From Eqs. 5.15, 5.18 and 5.25, the moment is

$$\underline{M} = - YI \frac{\partial \theta}{\partial x} = - YI \left(\frac{\partial^2 \underline{w}}{\partial x^2} + \frac{\Gamma \mu \omega^2}{YS} \underline{w} \right) . \tag{5.103}$$

Using Eqs. 5.42, 5.59, 5.67 and 5.68, the coefficient of $\underline{w}$ can be expressed in terms of δ, k_f and γ by

$$\frac{\Gamma \mu \omega^2}{YS} = 2\delta \frac{\omega \Omega}{c_\ell^2} = 2\delta k_f \gamma = k_f^2 - \gamma^2 . \tag{5.104}$$

Hence,

$$\underline{M} = - YI \left(\frac{\partial^2 \underline{w}}{\partial x^2} + (k_f^2 - \gamma^2) \underline{w} \right) . \tag{5.105}$$

The second spatial derivative of Eq. 5.102 is

$$\frac{\partial^2 \underline{w}}{\partial x^2} = - k_f^2 \underline{A} \, e^{-ik_f x} + \gamma^2 \underline{C} e^{-\gamma x} . \tag{5.106}$$

Substituting Eqs. 5.102 and 5.106 into Eq. 5.105 and setting the moment at $x = 0$ equal to 0 yields a relation between the two complex coefficients $\underline{A}$ and $\underline{C}$,

$$- k_f^2 \underline{A} + \gamma^2 \underline{C} + (k_f^2 - \gamma^2) (\underline{A} + \underline{C}) = 0 , \tag{5.107}$$

from which

$$\underline{A} = \frac{\gamma^2 + (k_f^2 - \gamma^2)}{k_f^2 - (k_f^2 - \gamma^2)} \underline{C} = \left(\frac{k_f}{\gamma} \right)^2 \underline{C} . \tag{5.108}$$

Since velocity at any point is related to displacement by the time derivative, it follows that

$$\dot{\underline{w}}(x) = i\omega \underline{w} = i\omega \underline{C} \left[\left(\frac{k_f}{\gamma} \right)^2 e^{-ik_f x} + e^{-\gamma x} \right] . \tag{5.109}$$

The expression for the force corresponding to this velocity is somewhat more complicated. From Eqs. 5.22, 5.25 and 5.105,

$$\underline{F} = \underline{F}_z = YI \left(\frac{\partial^3 \underline{w}}{\partial x^3} + (k_f^2 - \gamma^2) \frac{\partial \underline{w}}{\partial x} \right) + \alpha' \rho_s I \omega^2 \left(\frac{\partial \underline{w}}{\partial x} + \frac{\Gamma}{YS} \underline{F}_z \right) . \tag{5.110}$$

After some manipulation, this becomes

$$\frac{\underline{F}}{YI} = \frac{\underline{w}''' + (1 + \overline{\alpha}) (k_f^2 - \gamma^2) \underline{w}'}{1 - 4\overline{\alpha}\delta^2} , \tag{5.111}$$

where each prime indicates a differentiation with respect to x. Carrying out the differentiations of Eq. 5.102 with $\underline{A}$ given by Eq. 5.108, an expression for a force at $x = 0$ is

$$\underline{F}(0) = YI\underline{C}k_f^2 \frac{ik_f\left(1 - 2\overline{\alpha}\delta \frac{k_f}{\gamma}\right) - \gamma\left(1 + 2\overline{\alpha}\delta \frac{\gamma}{k_f}\right)}{1 - 4\overline{\alpha}\delta^2} . \tag{5.112}$$

The required force appears to become very large as the denominator approaches zero. However, the magnitude of the numerator also becomes zero and the ratio remains finite. Just as was done when deriving the flexural wave speed in Section 5.4, a close approximation for the force can be derived by ignoring the terms involving rotatory inertia. Setting $\overline{\alpha} = 0$, Eq. 5.112 becomes

$$\underline{F}(0) = YI\underline{C}k_f^2 \left(ik_f - \gamma\right) . \tag{5.113}$$

Dividing the force by the velocity at the end, $x = 0$, the input impedance is

$$\underline{Z}_i \equiv \frac{\underline{F}(0)}{\underline{\dot{w}}(0)} = \frac{YI\underline{C}k_f^2\left(ik_f - \gamma\right)}{i\omega\underline{C}\left(\left(\frac{k_f}{\gamma}\right)^2 + 1\right)} = \frac{YI}{\omega} k_f\gamma \frac{k_f + i\gamma}{\frac{k_f}{\gamma} + \frac{\gamma}{k_f}} = \mu v_f \frac{1 + i\left(\frac{\gamma}{k_f}\right)}{1 + \left(\frac{\gamma}{k_f}\right)^2} . \tag{5.114}$$

where the ratio of γ to k_f is given in Eq. 5.68 as a function of δ.

At low frequencies, i.e., for $\delta < 0.05$, the input impedance reduces to

$$\underline{Z}_{i_\ell} = \mu v_{f_\ell} \frac{1 + i}{2} . \tag{5.115}$$

Thus, at low frequencies the resistive and reactive components of the input impedance are equal. As frequency increases and effects of shear distortion are felt, the resistive term becomes somewhat larger and the reactive term smaller. At very high frequencies, $\delta > 3$, the reactive term is negligible and v_f in Eq. 5.114 can be replaced by v_{f_h}.

The power transferred to a semi-infinite beam by a velocity source is given by

$$W_i = R_i\overline{\dot{w}_o^2} = \frac{\mu v_f}{1 + \left(\frac{\gamma}{k_f}\right)^2} \overline{\dot{w}_o^2} , \tag{5.116}$$

while that transferred by a force source is

$$W_i = \frac{R_i}{|Z_i|^2} \overline{F^2} = \frac{1}{\mu v_f} \overline{F^2} . \tag{5.117}$$

This simple result is valid over the entire frequency range. Effects of shear at higher frequencies are incorporated in the expression for the flexural wave speed.

Infinite Beams

A beam may be considered infinite if the excitation occurs far enough from the ends that reflected energy is negligible. The derivation of the input impedance for this case is actually somewhat simpler than that for a semi-infinite beam. The force generates waves that progress away from the point of application, $x = 0$, in both directions. Thus, for $x \geqslant 0$,

$$\underline{w}(x \geqslant 0) = \underline{A}\, e^{-ik_f x} + \underline{C}\, e^{-\gamma x} \quad . \tag{5.118}$$

At the point of application of the force, the beam moves straight up and down without rotation. The boundary condition is therefore $\theta(0) = 0$. From Eq. 5.25, it follows that $w'(0) = 0$. Taking the derivative of Eq. 5.118 and setting it equal to zero, one finds

$$\underline{A} = i \frac{\gamma}{k_f} \underline{C} \quad , \tag{5.119}$$

from which it follows that the velocity is

$$\underline{\dot{w}}(x \geqslant 0) = i\omega \underline{C} \left(i \frac{\gamma}{k_f} e^{-ik_f x} + e^{-\gamma x} \right) \quad . \tag{5.120}$$

Since the applied force creates waves progressing in both directions, it must be twice as large as that required to create only the positive-direction wave. Setting θ and w' equal to zero, Eqs. 5.15, 5.18, 5.22 and 5.25 yield

$$\underline{F} = -2 \frac{\partial \underline{M}}{\partial x} = 2YI \frac{\partial^2 \underline{\theta}}{\partial x^2} = 2YI \frac{\partial^3 \underline{w}}{\partial x^3} \quad . \tag{5.121}$$

Taking the third derivative of Eq. 5.118, with $\underline{A}$ given by Eq. 5.119, Eq. 5.121 becomes

$$\underline{F} = -\ 2YI\underline{C}\gamma\,(k_f^2 + \gamma^2) = -\ 4YI\underline{C}\gamma k_{f_o}^2 \sqrt{1 + \delta^2} \quad , \tag{5.122}$$

from which the input impedance is

$$\underline{Z}_i = \frac{F}{\underline{\dot{w}}_o} = 2\mu v_f \left(1 + i \frac{k_f}{\gamma} \right) \quad . \tag{5.123}$$

At low frequencies, this is four times as large as that for a semi-infinite beam, Eq. 5.115. As the frequency increases, the reactive term increases more rapidly than the resistive term. The magnitude of the impedance therefore continues to increase with frequency rather than approaching a constant value, as does that of a semi-infinite beam.

The power transferred by a velocity source to an effectively infinite beam is

$$W_i = R_i \overline{w_o^2} = 2\mu v_f \overline{\dot{w}_o^2} \quad , \tag{5.124}$$

while that received from a force is

$$W_i = \frac{R_i \overline{F^2}}{|Z_i|^2} \doteq \frac{1 - \dfrac{\delta}{\sqrt{1 + \delta^2}}}{4\mu v_f} \overline{F^2} . \tag{5.125}$$

The latter expression shows that away from their ends beams become increasingly resistant to the absorption of power from applied forces as frequency increases. It was this fact that led to the statement in Section 5.4 that flexural waves are not usually dominant at the higher frequencies, i.e., for $\delta > 3$.

Role of Damping

The criterion for a finite beam to behave as an infinite beam is that reflected energy be negligible. This will occur if power is absorbed at an end or if the vibration is sufficiently damped in traveling from the source to the end. Every flexing beam experiences at least a little damping. The alternating extensional motions of its fibers involve storing and release of energy, which process invariably involves energy dissipation.

In dealing with linear systems at a fixed frequency, energy dissipation can be incorporated into the analysis by replacing certain real quantities with complex ones, the imaginary components of which are proportional to the dissipation. This was done in Section 2.4 in considering damped sound waves in slightly lossy fluids and in Section 4.3 relevant to pulsating bubbles. In treating beam flexural vibrations, the elastic moduli Y and G control energy storage and therefore account for dissipation. Since the two moduli are related through Poisson's ratio by Eq. 5.5 and that ratio is usually real, the same loss factor applies to both. The procedure used is to replace Y by $Y(1 + i\eta)$ and G by $G(1 + i\eta)$ wherever they occur. When this is done,

$$c_\ell \rightarrow c_\ell \sqrt{1 + i\eta} \doteq c_\ell \left(1 + i\frac{\eta}{2}\right) , \tag{5.126}$$

$$v_{f\ell} \rightarrow v_{f\ell} \sqrt[4]{1 + i\eta} \doteq v_{f\ell} \left(1 + i\frac{\eta}{4}\right) , \tag{5.127}$$

$$\delta \rightarrow \frac{\delta}{\sqrt{1 + i\eta}} \doteq \delta \left(1 - i\frac{\eta}{2}\right) , \tag{5.128}$$

and

$$k_{f_o} = \frac{\omega}{v_{f\ell}} \rightarrow k_{f_o} \left(1 - i\frac{\eta}{4}\right) . \tag{5.129}$$

The effect of damping on input impedances of non-resonant structures is usually quite small. In effect, the out-of-phase components are altered slightly. Thus, neglecting secondary effects, the input impedance of a semi-infinite beam becomes

$$\underline{Z}_i \doteq \mu v_f \frac{1 + i\left(\dfrac{\gamma}{k_f} + \dfrac{\eta}{4}\right)}{1 + \left(\dfrac{\gamma}{k_f}\right)^2} \quad . \tag{5.130}$$

Unless $\eta > 0.5$, the effect of damping is quite small.

A more important effect of damping is that it causes spatial decay of the vibrations. Without damping, the vibratory motion would be the same at all positions on a non-resonant beam remote from the source. The effect of damping is to introduce attenuation. Substituting Eq. 5.129 for k_f in Eq. 5.109, the velocity at distance x becomes

$$\underline{\dot{w}} = i\omega \underline{C} \left(\frac{k_f}{\gamma}\right)^2 e^{-(\eta/4)k_f x} \, e^{i(\omega t - k_f x)} \quad . \tag{5.131}$$

Thus, the space-rate-of-decay of the vibration is

$$-\frac{1}{\dot{w}} \frac{d\dot{w}}{dx} = \frac{1}{4} \eta k_f \doteq 13.65\eta \quad dB/wavelength \; . \tag{5.132}$$

A beam can be treated as effectively infinite if the source is located distance $1/2\eta$ flexural wavelengths from the nearest reflecting termination.

Damping also controls the rate at which a vibration will decay once the source is removed. Thus, it can readily be shown that, upon securing a source, a flexural vibration will decay at a rate given by

$$-\frac{1}{\dot{w}} \frac{d\dot{w}}{dt} = \frac{1}{2} \eta\omega = 27.3\eta f \quad dB/sec \; . \tag{5.133}$$

Damping is often measured by finding the time-rate-of-decay of vibration.

5.8 Forced Vibrations of Resonant Structures

Role of Resonances

As discussed in Section 5.5, finite beams are multiply-resonant systems. When allowed to vibrate freely, finite beams vibrate at one or more of their resonance frequencies. These resonances also play an important role in the beam's response to an applied force, acting as *modes* or *eigen-frequencies*. Response of a beam to applied forces can be calculated as the sum of its responses at all of the eigen-frequencies. Skudrzyk (1958 and 1968) has based his extensive treatment of this subject on the fact that any lumped or homogeneous system can be represented by an infinite number of series-resonant circuits all connected in parallel, each of which represents one mode. Obviously, the closer the exciting frequency is to the resonant frequency of a mode, the greater its excitation, provided the force is not applied at a spatial node of that mode.

Usually one is interested in a structure over a range of frequencies. The approach taken is to plot its calculated or measured impedance or admittance as a function of frequency. Resonances and/or anti-resonances will then show clearly and the suitability of the system for its intended application should be clear. Skudrzyk (1968) and Snowdon (1968) have shown that input impedance and admittance functions are characterized by alternating resonances and anti-resonances, while transfer functions sometimes exhibit only resonances.

Modal Responses

The total impedance of a number of parallel circuits is the reciprocal of the sum of their admittances. In most cases, the location and nature of the source is such that the different modes experience different amounts of excitation. The total response of the structure is then the sum of the modal responses weighted by the excitation of each. For an infinite number of undamped modes,

$$\dot{w} \sim \frac{F}{\mu L} \sum_{m=1}^{\infty} \frac{\omega \phi_m}{\omega_m^2 - \omega^2} , \tag{5.134}$$

where ϕ_m is a weighting function giving the relative force for each mode. Without damping, the response is infinite at each resonance.

In resonant systems, material damping has the important role of limiting resonant and anti-resonant responses. From Eqs. 5.63 and 5.126,

$$\omega_m \rightarrow \omega_m \left(1 + i \frac{\eta}{2}\right) , \tag{5.135}$$

and the mean-square velocity is

$$\overline{\dot{w}^2} \sim \frac{\overline{F^2}}{\mu^2 L^2} \sum_{m=1}^{\infty} \frac{\omega^2 \phi_m^2}{\left(\omega_m^2 - \omega^2\right)^2 + \eta^2 \omega_m^4} . \tag{5.136}$$

The half-power points on the resonance curve occur when the two terms in the denominator of Eq. 5.136 are equal, i.e., at

$$\left| \omega_m - \omega \right| \doteq \frac{\eta}{2} \omega_m . \tag{5.137}$$

It follows that the sharpness of the resonance, as given by its Q, is

$$Q \equiv \frac{f}{\Delta f} = \frac{\omega_m}{2 \left| \omega_m - \omega \right|} = \frac{1}{\eta} . \tag{5.138}$$

It can also be shown that at low frequencies the amplitude of the motion occurring at a resonance is Q times that which would occur if the system were non-resonant. Also, at low-frequency anti-resonances the impedance is Q times that for the non-resonant system. Thus, the input impedance of a structure treated as non-resonant equals the geometric mean of the values at resonances and anti-resonances.

Broadband Excitation

In many instances the exciting force covers a band of frequencies that is wide compared to the bandwidths of any resonances within the band. In this case, the response of each resonance is obtained by integrating over frequency across the resonance. Thus

$$\overline{\dot{w}_m^2} \sim \frac{\overline{F^2}\,\phi_m^2}{\mu^2 L^2(\omega_2 - \omega_1)} \int_{\omega_1}^{\omega_2} \frac{\omega^2\, d\omega}{(\omega_m^2 - \omega^2)^2 + \eta^2\omega^4}$$

$$\doteq \frac{\overline{F_m^2}}{\mu^2 L^2(\omega_2 - \omega_1)} \, \frac{\pi}{2\eta\omega_m} \, , \tag{5.139}$$

provided $(\omega_2 - \omega_1)$ is large compared to the width of the resonance. The power accepted by this resonance is given by Eq. 5.100 as

$$W_m = R\overline{\dot{w}_m^2} = \eta\omega\mu L \cdot \overline{\dot{w}_m^2} \doteq \frac{\pi}{2} \, \frac{\overline{F^2}\,\overline{\phi_m^2}}{\mu L(\omega_2 - \omega_1)} \, . \tag{5.140}$$

If the density of resonances is $dN/d\omega$, then there will be

$$N = \frac{dN}{d\omega}(\omega_2 - \omega_1) \tag{5.141}$$

resonances within the band, and the total power will be

$$W_i \doteq \frac{\pi}{2} \, \frac{\overline{F^2}}{\mu L} \, \frac{dN}{d\omega} \, , \tag{5.142}$$

provided each resonance is excited equally.

The expression of power transferred to a resonant structure is seen to depend only on the modal density divided by the total effective mass. Following Nelson (1972), we may write

$$\frac{dN}{d\omega} = \frac{dN}{dk_f}\,\frac{dk_f}{d\omega} = \frac{1}{\mathrm{v}_f}\,\frac{dN}{dk_f}\left(1 - k_f\frac{d\mathrm{v}_f}{d\omega}\right) = \frac{L}{\pi \mathrm{v}_f}\left(1 - k_f\frac{d\mathrm{v}_f}{d\omega}\right) . \tag{5.143}$$

and the input power becomes

$$W_i \doteq \frac{\overline{F^2}}{2\mu \mathrm{v}_f}\left(1 - k_f\frac{d\mathrm{v}_f}{d\omega}\right) . \tag{5.144}$$

At low frequencies, the dispersion term is 1/2 and the power reduces to

$$W_{i\ell} = \frac{\overline{F^2}}{4\mu \mathrm{v}_{f\ell}} \, . \tag{5.145}$$

in complete agreement with that for an infinite beam, as given by Eq. 5.125. It follows that the response of a resonant structure to wideband excitation is the same as that of a non-resonant structure having the same parameters.

5.9 Attenuation of Structural Vibrations

In Section 1.1 it was noted that reducing the efficiency of vibration transmission from a source to a radiating surface is usually the easiest way of achieving noise reduction. There are a number of ways of attenuating such structural vibrations, several of which will be discussed briefly in this section. Readers desiring more information are referred to the extensive list of references on this subject at the end of this chapter.

Isolation Mounts

The most common method of reducing structural vibrations is to interpose a relatively flexible vibration isolator between a source of vibrations and a structural member. The force generated by a machine, F_i, normally would act to cause both the machine and its foundation to vibrate. If the two are rigidly connected, they must share the same velocity and the input force must be divided between them in proportion to their impedances. The force transmitted to the foundation is therefore

$$\underline{F}_f = \frac{F_i \underline{Z}_f}{\underline{Z}_s + \underline{Z}_f} , \tag{5.146}$$

where $\underline{Z}_f$ is the foundation impedance and $\underline{Z}_s$ is the internal impedance of the source. If we now interpose an isolator with impedance $\underline{Z}_i$ between source and foundation, the isolator and foundation will share the same force but divide the velocity. In other words, they will act as parallel impedances. The force imparted to the foundation will then be given by

$$\underline{F}_f^i = \frac{F_i \dfrac{\underline{Z}_f \underline{Z}_i}{\underline{Z}_i + \underline{Z}_f}}{\underline{Z}_s + \dfrac{\underline{Z}_f \underline{Z}_i}{\underline{Z}_i + \underline{Z}_f}} = \frac{F_i \underline{Z}_i \underline{Z}_f}{\underline{Z}_i \underline{Z}_s + \underline{Z}_f \underline{Z}_s + \underline{Z}_f \underline{Z}_i} . \tag{5.147}$$

The effectiveness of the mount is defined as the ratio of non-isolated to isolated foundation forces and is given by

$$e \equiv \left| \frac{\underline{F}_f}{\underline{F}_f^i} \right| = \left| \frac{\underline{Z}_i \underline{Z}_s + \underline{Z}_f \underline{Z}_s + \underline{Z}_f \underline{Z}_i}{\underline{Z}_i (\underline{Z}_f + \underline{Z}_s)} \right| . \tag{5.148}$$

This expression is somewhat simpler if instead of the impedances of the elements one uses their mobilities, or admittances. Thus

$$e = \left| \frac{\underline{Y}_f + \underline{Y}_s + \underline{Y}_i}{\underline{Y}_f + \underline{Y}_s} \right| . \tag{5.149}$$

The logarithmic expression for mount effectiveness is called *insertion loss*. Insertion loss measures the value of an isolator as a noise reduction device. It is more meaningful than the often measured transmission loss, which is the ratio of the vibratory motion of the source to that of the foundation. This ratio is given by

$$t \equiv \left| \frac{\dot{\underline{w}}_s}{\dot{\underline{w}}_f} \right| = \left| \frac{\underline{Z}_i + \underline{Z}_f}{\underline{Z}_i} \right| = \left| \frac{\underline{Y}_i + \underline{Y}_f}{\underline{Y}_f} \right| , \qquad (5.150)$$

and is always larger than the insertion loss. It only measures whether an isolator is operational, i.e., whether $\underline{Z}_i << \underline{Z}_f$ as it must be for maximum effectiveness.

The expressions for mount effectiveness given by Eqs. 5.148 and 5.149 are general. We can better understand how mounts work by considering some special cases. The simplest system is one in which the machine is a mass, the isolator is a lossy spring and the foundation presents infinite impedance. In this ideal case,

$$e_i \doteq \left| 1 - \left(\frac{f}{f_o} \right)^2 (1 - i\eta) \right| , \qquad (5.151)$$

where f_o is the resonance frequency of source mass and isolator spring constant, and η is the loss factor of the system, assumed less than 0.1. As shown in Fig. 5.14, the mount is ineffective at low frequencies. In fact, at resonance, it serves to magnify the transmitted force by an amount that is only limited by damping. Well above resonance, the effectiveness of an ideal mount increases by 12 dB/octave.

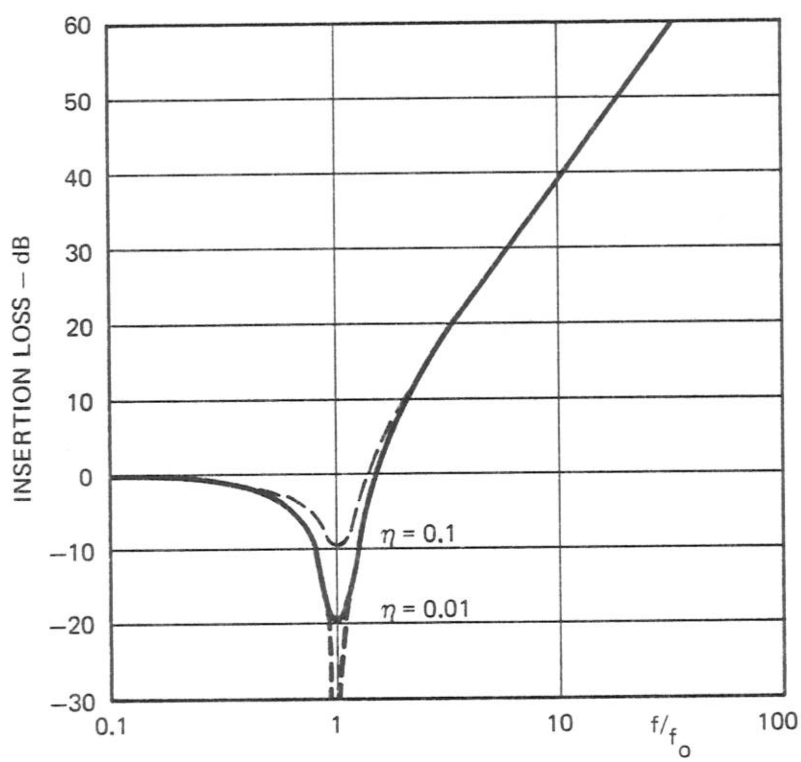

Fig. 5.14. Insertion Loss of an Ideal Isolation Mount

In the ideal case, insertion loss increases indefinitely with increasing frequency and the optimum mount is the one with the lowest resonant frequency. In practical shipboard systems, however, actual insertion losses seldom exceed 30 dB and 10 to 20 dB are typical. There are a number of reasons for this departure from the ideal. First, springs used as isolators are themselves distributed systems which may have resonances at high frequencies. At these resonances, their mobilities are very small and mount effectiveness, by Eq. 5.149, is close to unity. It is to avoid such wave effects that most modern isolators are composed extensively of rubber, used either in compression or in shear. A second reason for less than ideal performance of mounts on ships is the relative mobility of the foundation. Foundations are generally composed of beams which, being finite, resonate at a number of frequencies. At such resonances the foundation impedance can become very small relative to that of the source itself. In such a case,

$$e \doteq \left| 1 + \frac{\underline{Z}_f}{\underline{Z}_i} \right| , \qquad (5.152)$$

and the mount effectiveness is controlled by the ratio of foundation to isolation impedances. At resonances, as discussed in Section 5.8, the impedance is entirely controlled by the resistive component. It is for this reason that Sykes (1958, 1960), Klyukin (1961), Ungar (1962) and others have recognized the importance of building extra damping into foundation structures. The final reason for less than ideal mount performance is that the machine itself also has resonances. With increasing frequency, its mobility tends to become constant on the average, rather than to decrease as it would if it were a pure mass. Since springs composed of rubber have almost constant mobility at high frequencies, the insertion loss tends toward a constant, limiting value.

Applied Damping

The importance of damping in limiting system responses at resonances has been stressed in both Section 5.8 and the discussion of vibration isolation mounts. As will be indicated in Chapter 6, damping also controls plate resonant vibrations and thereby affects sound radiation. For instance, mastic undercoat is used on automobiles and railway cars to reduce their resonant responses and make them sound less *tinny*. In fact, development of damping for plates and structural members has been one of the more active areas of noise control development over the past 30 years.

One approach to damping has been the development of a number of structural materials having high internal damping. In his review of this subject, Adams (1972) noted several materials having favorable damping characteristics. However, these materials are quite expensive and most research in this area has focused instead on ways of damping ordinary metal structures. Many rubbers have high internal damping and much of the research has concentrated on the development of rubber-like (viscoelastic) materials that can be sprayed on or otherwise readily attached to metal. Oberst (1952, 1954, 1956) and his co-workers in Germany have developed chemical methods to produce such damping materials.

Oberst analyzed the damping of plates by *homogeneous layers* of damping material, attributing the damping to the extensional-compressional motion which these layers experience as the structure flexes. Consider the single-layer treatment sketched in Fig. 5.15. The solid rectangular base is characterized by Young's modulus Y and thickness H_1. The viscoelastic layer has thickness H_2 and its Young's modulus, which includes damping, is $Y_2(1 + i\eta_2)$. The neutral plane of the combined

Fig. 5.15. Homogeneous Damping Treatment

plate is displaced z_o from the centerline of the base section due to the additional layer. The bending rigidity per unit width of the combined structure is given by

$$\underline{B} = B(1 + i\eta) = \sum \underline{Y}_i I_i = Y_1 \frac{H_1^3}{12} + Y_1 H_1 z_o^2 + \underline{Y}_2 \frac{H_2^3}{12} + \underline{Y}_2 H_2 \left(\frac{H_1 + H_2}{2} - z_o \right)^2 . \tag{5.153}$$

The displacement of the neutral plane can be found from the requirement that the net extensional force be zero; thus,

$$Y_1 H_1 z_o = Y_2 \sqrt{1 + \eta^2} H_2 \left(\frac{H_1 + H_2}{2} - z_o \right) . \tag{5.154}$$

Assuming the extensional stiffness of the damping layer, $Y_2 H_2$, to be small compared to that of the base, $Y_1 H_1$,

$$z_o \doteq \frac{Y_2 H_2}{Y_1 H_1} \sqrt{1 + \eta^2} \left(\frac{H_1 + H_2}{2} \right) . \tag{5.155}$$

Substituting Eq. 5.155 into Eq. 5.153, the effective damping factor of the combination is

$$\eta \doteq \eta_2 \frac{Y_2 H_2}{Y_1 H_1} \frac{3H_1^2 + 6H_1 H_2 + 4H_2^2}{H_1^2 + \frac{Y_2 H_2}{Y_1 H_1} (3H_1^2 + 6H_1 H_2 + 4H_2^2)} . \tag{5.156}$$

For many cases, this reduces to

$$\eta \doteq 3\eta_2 \frac{Y_2 H_2}{Y_1 H_1} \left(1 + \frac{H_2}{H_1} \right)^2 . \tag{5.157}$$

showing that the damping is proportional to the product of the loss coefficient of the material and the extensional thickness of the damping layer, magnified by a factor that represents the relative separation of the centers of the two layers.

Oberst's results are plotted in Fig. 5.16. For very thin layers, the dependence on relative thickness is linear, but the resultant loss factor is less than 0.01. If a damping treatment is to be really useful, it should produce a loss factor of at least 0.05, which for most materials requires a thickness of treatment of the same order as that of the base. Ross, Ungar and Kerwin (1959) showed that, using the best damping materials then available, a thickness ratio of 1.25 was required to achieve $\eta = 0.1$ on steel structures, and a ratio of 0.7 on aluminum ones. These thicknesses correspond to weight ratios of 16% for steel and 28% for aluminum.

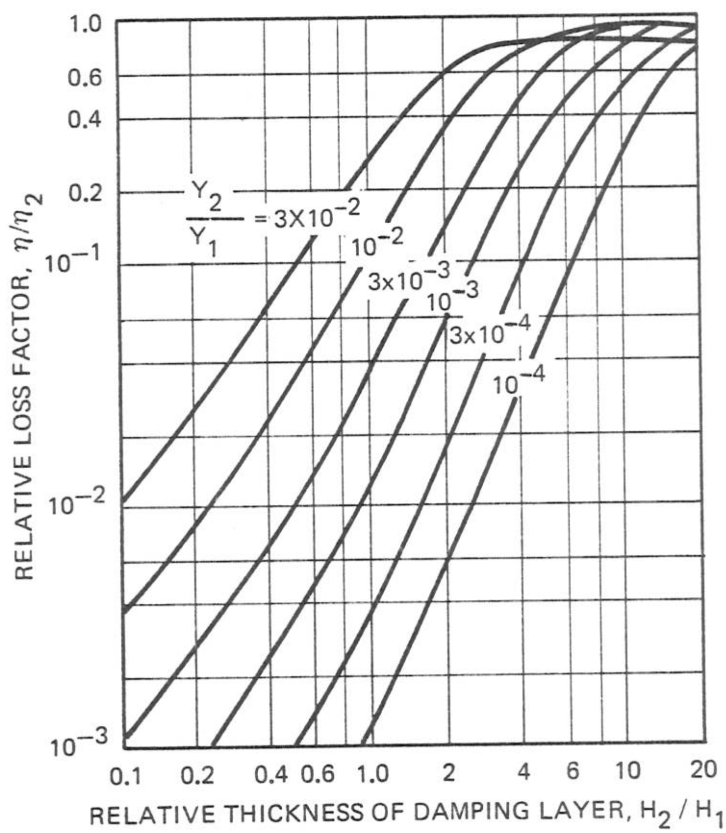

Fig. 5.16. Relative Damping as a Function of Layer Thickness, after Oberst (1952)

The problem with homogeneous damping treatments is that they are relatively heavy and bulky. They were developed for use on relatively light plates and are of little use on beams. Kerwin (1959) observed that the use of a thin metal cover on top of a damping layer causes the latter to experience shear and that such shearing action can be more efficient in producing damping. He derived an expression for the loss factor of *constrained-layer damping treatments* and verified the results experimentally. Ross, Ungar and Kerwin (1959) published formulas for optimized constrained-layer treatments, finding that the same weight of treatment produces from two to four times as much damping as that of a homogeneous layer. Significant damping can be achieved with treatments weighing less than 8% as much as the base. If very high damping is desired, a sandwich can be built in which two equal metal bars or plates are separated by a thin viscoelastic layer, as described by Kurtze (1959). In this case, the resultant loss factor is about 25% of that of the damping material.

As found by Kerwin, Ross and Ungar, shear damping treatments are more frequency and/or temperature dependent than are homogeneous, extensional types. Figure 5.17 shows the frequency dependence of a typical shear treatment at two temperatures. There are several ways of

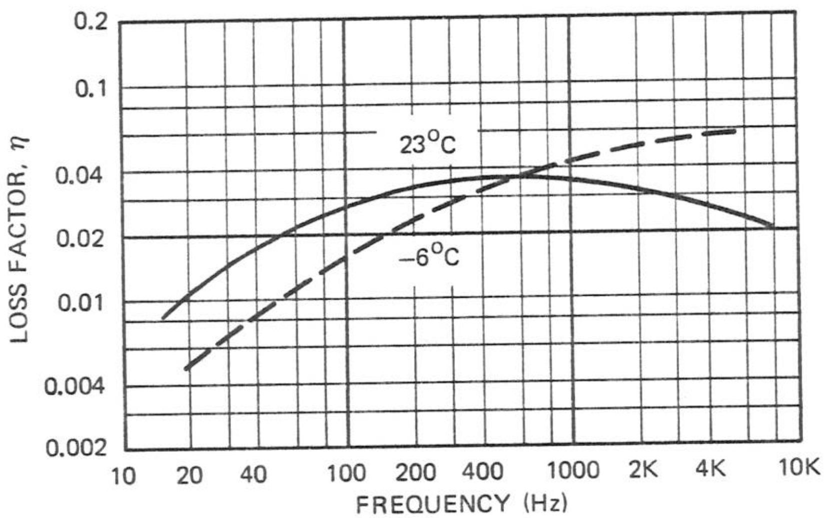

Fig. 5.17. Frequency and Temperature Dependencies of a Typical Shear Damping Treatment, from Ross, et al (1959)

overcoming this problem. Ungar and Ross (1959) analyzed multiple-layer treatments and found that increasing the number of layers broadens the peak region, as shown in Fig. 5.18. Grootenhuis (1970) has developed treatments in which two different viscoelastic materials are used under a single constraining layer, achieving significant broadening of the region of high damping.

A major advantage of shear damping is its applicability to beam structures. Ruzicka (1961) and Ungar (1962) have developed and evaluated a number of different ways of incorporating damping in beams, some of which are shown in Fig. 5.19.

Impedance Mismatches

Another approach to the attenuation of flexural waves is the introduction of changes of cross section and the attachment of mass elements, all of which create impedance mismatches which act to reflect a fraction of an incident flexural wave. Cremer (1953, 1956) analyzed cross-sectional changes, finding a transmission loss of 3 dB for a 5:1 ratio of section thicknesses, which increases about 4 to 5 dB for every doubling of this ratio. Rader and Mao (1971) have analyzed this case by analogy to Snell's law. When bending waves are made to turn a corner, a 3 dB reduction occurs. This can be increased by simultaneously changing the structural rigidity. Cremer also found that the attachment of a concentrated mass load to a beam may produce a change in moment of inertia

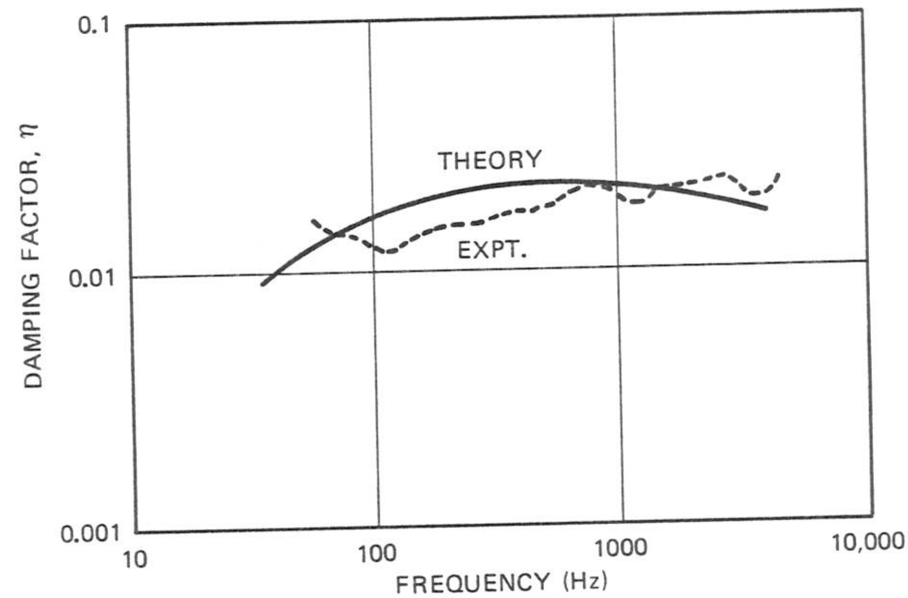

Fig. 5.18. Damping Curve for a Double-Layer Damping Treatment Having a Total Weight of 5% of the Base Plate, from Ungar and Ross (1959)

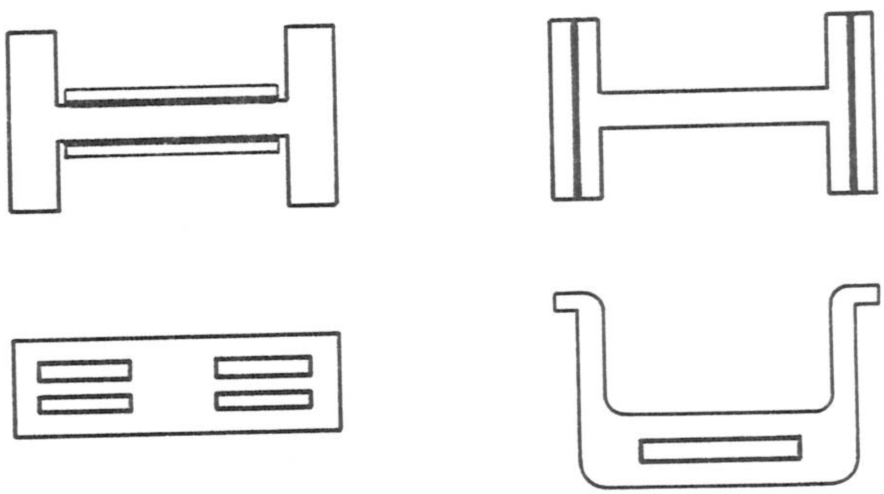

Fig. 5.19. Examples of Damped Beam Structures, after Ruzicka (1961) and Ungar (1962)

sufficient to cause as much as 20 to 30 dB of attenuation above a minimum frequency. Below this frequency, the mass acts as though distributed and the loss is negligible.

Periodically-spaced impedance discontinuities in the form of attached masses are not as effective as might be expected. The reason is that the structure itself develops resonances with nodes at the attachments. Waves at these frequencies are passed without attenuation. Mead (1970) and Sen

Gupta (1970) have analyzed periodic structures by methods similar to those developed by Brillouin (1953) to analyze energy propagation in crystal lattices. They defined a complex propagation constant which is sometimes real, corresponding to attenuation, and which is imaginary in certain frequency bands that pass energy virtually unattenuated. Obviously, multiple attachments are best used with irregular spacings, so that there will be significant attenuation at all frequencies above the low-frequency limit.

Vibration Absorbers and Suppressors

Yet another way of attenuating flexural waves is by attaching devices to a vibrating structure that will either absorb the energy or feed back a cancelling signal that suppresses the vibration. Klyukin (1960), having noted that vibration-sensing instruments act to suppress the vibrations they are intended to measure, proposed a number of different passive vibration absorbers. These attachments, which are resonant, consist of masses on springs with dampers. Such systems having Q's as low as 2 can provide as much as 40 dB of attenuation over an octave. Machinery on isolation mounts located inside ships undoubtedly contribute to the damping of hull flexural vibrations by just this mechanism.

Active electromechanical feedback vibration suppressors have been developed and tested by Knyazev and Tartakovskii (1965, 1966 and 1967). The dispersive nature of flexural waves makes such an approach more difficult than when the wave speed is constant, but this difficulty was overcome with a phase-compensating feedback system. Attenuations at resonances of the order of 15 dB were achieved. In a parallel development, Rockwell and Lawther (1964) demonstrated similar reductions for a uniform beam supported by rubber mounts, using a co-located sensor and feedback source. In principle, and with sufficient investment, active dampers would be very effective.

5.10 Fluid Loading

Immersion of a structure in a relatively dense fluid such as water can change its vibrational characteristics significantly. As compared to vibrations in air, the effective mass of the structure is increased by the mass of the entrained fluid, and both fluid viscosity and the radiation of sound add to the damping. Of these effects, the first two are more important for beams, while the last is a major consideration for plates (see Chapter 6).

Entrained Mass

Entrained mass has been accounted for in the derivations given in Sections 5.3-5.8 by inclusion of the relative entrained mass, ϵ, defined as the ratio of entrained mass to that of the structure. However, its significance was not evaluated, nor were methods for its calculation discussed. In many instances, such as heavy foundation structures, the relative entrained mass is very small. However, in the case of neutrally buoyant structures, such as ship hulls, the entrained mass may exceed the structural mass. From Eqs. 5.65 and 5.115 it is apparent that low-frequency resonance frequencies are inversely proportional to $\sqrt{1+\epsilon}$ and that flexural impedances increase linearly with total mass. Accurate calculation of entrained mass therefore becomes increasingly important as its magnitude increases, and it is not surprising that naval architects have given extensive attention to this subject. At these frequencies, compressibility of the fluid is unimportant and classical incompressible hydrodynamics theory can be used.

As derived by Morse (1948) and others, the entrained mass of an infinitely long, rigid, cylindrical rod oscillating in a plane equals that of the displaced fluid. Thus, for a rigid cylinder,

$$\epsilon = \frac{\rho_o}{\rho_s} \quad . \tag{5.158}$$

However, actual structures are neither cylindrical, infinite nor rigid. Lewis (1929) assumed vibrational mode shapes to be the same as those in air and calculated the entrained mass for slender bodies of circular, rectangular and ship-like cross sections in incompressible fluids. His results are usually written in the form

$$m_e = \rho_o \frac{\pi b^2}{4} J_m(b/L) C(b/h, S/bh) \quad , \tag{5.159}$$

where J_m accounts for finite flexural wavelengths and C is a shape factor which equals unity for circular and elliptical cross sections. Chertock (1975) has obtained Lewis' results by a simpler formulation derived from the Helmholtz integral, Eq. 4.140. Townsin (1969) has matched other theoretical and measured values of J for various order modes by

$$J_m \doteq 1.02 - 3\frac{b}{L}\left(1.2 - \frac{1}{m}\right) \quad . \tag{5.160}$$

Because of the complexity of ship structures and of their resultant vibrations, simple formulas for entrained mass are not likely to be accurate. On the other hand, values within ±10% are sufficiently accurate for most purposes.

Hydrodynamic Damping

Damping due to fluid viscosity depends on section shape, amplitude of the motion, and the steady-state flow speed. Sharp edges increase this damping. Blake and Maga (1975) found values of the hydrodynamic loss factor to be from 10^{-2} to 10^{-1} for struts in water. Further discussion of this topic is beyond the scope of the present volume.

Sound Radiation

As long as the cross-sectional circumference of a submerged beam is small, each section will radiate sound as an unbaffled piston. As discussed in Section 4.8, unbaffled structures radiate as dipoles at low frequencies, with radiation efficiencies proportional to $(ka_o)^3$. Since the various sections of a long beam vibrate out of phase, the radiation efficiency for beam vibrations should be even less than that for a free piston.

Yousri and Fahy (1973) and Kuhn and Morfey (1974) have calculated the sound radiated by a uniform beam, finding a strong dependence on both the aspect ratio of the beam and the ratio of the flexural wave speed to the speed of sound. Their results have been confirmed by experiments reported by Blake (1974). His data show $\eta_{rad} < 10^{-3}$ when $v_f < 0.5c_o$. In view of these results, one can conclude that radiation damping of structural vibrations is generally negligible for beam-like structures in comparison with hydrodynamic damping.

5.11 Flexural Resonances of Ship Hulls

Although the sound radiated is negligible, resonant flexural vibrations of ship hulls are of vital importance to naval architects both because of potential damage to the structure and because of adverse responses of humans exposed to such vibrations. The problem faced by a ship designer is to estimate both the driving and resonance frequencies and to take steps to prevent their coincidence.

The methods discussed in Section 5.6 pertaining to the calculation of resonance frequencies of non-uniform beams are all applicable to this problem. Using the terminology used earlier in this chapter, ship hulls are characterized by very large values of the shear parameter, Γ, and much smaller values of the relative rotatory inertia coefficient, α'. Thus, examples of ship structures described by McGoldrick and Russo (1955) and Andersson and Norrand (1969) have values of Γ greater than 200, while α' seldom exceeds 10. The reason for the high Γ is that the effective shear-carrying area of a ship is a very small fraction of the total cross section.

There are two important practical consequences of the high values of Γ found for ships. First, shear effects are experienced even at the lowest resonance, and calculations which ignore shear are seriously in error. Secondly, since $\alpha' << \Gamma$, it is quite safe to ignore rotatory inertia when calculating ship flexural motions. Thus, the approximations involved in Sections 5.4 and 5.7 in calculating flexural wave speeds and input impedances, in which α' was set equal to zero, are especially valid.

Methods used by naval architects to calculate flexural resonances of ships are described by Leibowitz and Kennard (1961) and by McGoldrick (1960). These include finite-element techniques involving both analog and digital computers. Generally, the ship is divided into 20 equal sections, each of which is assumed to form a Timoshenko beam element. Also, there are more than a dozen semi-empirical formulas which can be used to find the fundamental frequency. The higher order modes can then be assumed to be linearly related to the fundamental. One especially simple formula for the fundamental, based on measured natural frequencies of commercial ships, relates this frequency inversely to the length by

$$f_{fund} \doteq \frac{A}{L} \ (Hz) \ , \tag{5.161}$$

where $A = 215$ if L is in meters and $A = 700$ if L is in feet.

The wave approach to resonance calculation described in Sections 5.5 and 5.6 would seem to be particularly well suited to this problem. The author has attempted to use this method to calculate the resonances of a specific ship with moderate success. The problem is to determine the phase shift caused by the ends. When the ends are each assumed to cause a 90° phase shift, as they would for a free-free uniform beam, frequencies calculated for the lowest order resonances are much too low. On the other hand, if it is assumed that the shear rigidity is zero at the ends and the phase shifts are zero, then these frequencies are somewhat too high. Based on this single attempt, it appears that the assumption of zero phase shift is more useful. However, more research needs to be done.

One advantage of the wave approach, as compared to finite-element methods, is that the ship can be divided into sections at natural boundaries. The flexural wave speed for each section can then be calculated and the total travel time found from

$$T = \sum_{i=1}^{N} \frac{\Delta L_i}{v_{f_i}} . \tag{5.162}$$

Using the symbology of naval architecture, the reciprocal of the flexural wave speed, Eq. 5.44, can be calculated from

$$\frac{1}{v_f} \doteq \left[\sqrt{\frac{\mu}{\omega^2 YI} + \left(\frac{\mu}{2KGS}\right)^2} + \frac{\mu}{2KGS} \right]^{1/2} . \tag{5.163}$$

Since v_f depends on frequency, the procedure used is to calculate T for four or five frequencies covering the likely range of resonances and to plot T vs ω or f. Since resonances occur when

$$T = \frac{m\pi + \overline{\Delta\phi}}{L\omega_m} , \tag{5.164}$$

this expression for T can also be plotted as a function of frequency for various values of m, and resonance frequencies are then those values for which the curves from Eq. 5.162 and 5.164 intersect.

It seems unlikely that more than five modes would occur with sufficient strength to be excited. For higher frequencies, attenuation due to both hydrodynamic damping and reflections at structural discontinuities would preclude the occurrence of flexural resonances that involve wave travel over the entire hull length. Instead, resonances involve vibrations of only part of the length of the ship. Such compartment resonances are usually dominant at frequencies above about 10 times that of the lowest flexural resonance.

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