CHAPTER 2

SOUND WAVES IN LIQUIDS

2.1 Description of Waves

A *wave* is an energy-carrying disturbance moving through a distributed medium. Familiar examples include surface waves on water, waves in strings and electromagnetic (radio) waves. Sound energy is carried by longitudinal waves, which involve alternating compressions and rarefactions of the medium. Sound waves occur in gases, liquids and solids. Derivations of the pertinent equations are slightly different for the three types of media, although the basic nature of the wave motion is the same. Discussions and derivations given in this book are specific to liquids, but apply almost equally well to gases. The equations for sound waves in solids are more complex (see Officer, 1958).

Figure 2.1 illustrates the simplest example of wave motion, such as that on a string. A simple disturbance moves along the x-axis with wave speed c, maintaining its shape as it progresses. The most general function describing such a motion is of the form $f(x - ct)$, which is a solution of the second-order differential equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0\ , \qquad (2.1)$$

as can readily be shown by performing the indicated partial differentiations. This equation is the *wave equation* for a wave progressing in the x direction.

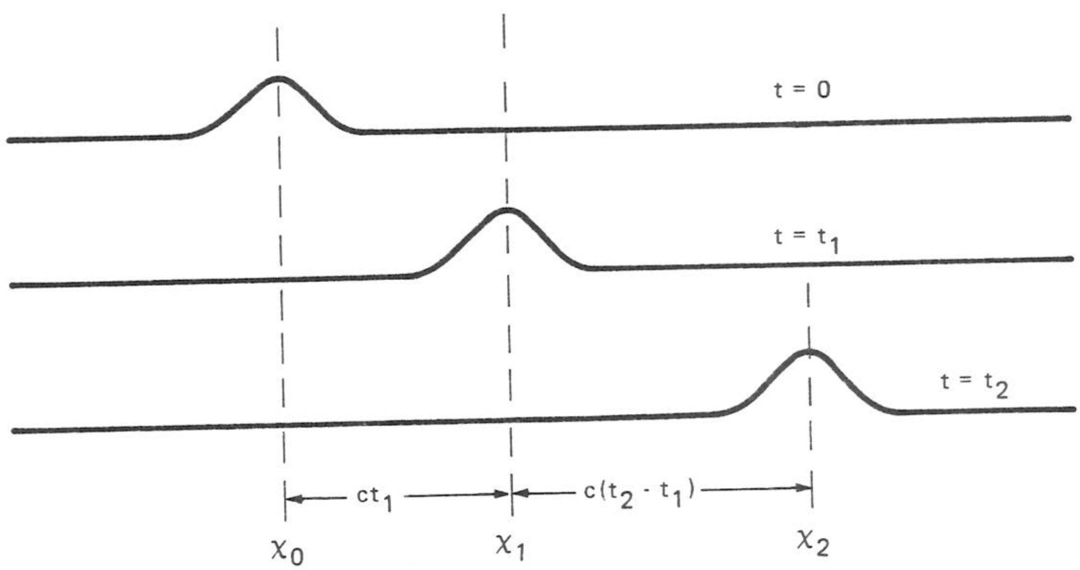

Fig. 2.1. A Simple Progressing Wave

Plane Waves

A general analytic expression for a plane wave in space is of the form $F(r - ct)$, where r is distance traveled by the wave in the direction of propagation. In cartesian coordinates

$$F(r - ct) = F(n_x x + n_y y + n_z z - ct) \ , \tag{2.2}$$

where the various n's are the three direction cosines, i.e., the three coordinate projections of a unit vector normal to a plane of constant phase. The direction cosines satisfy the relationship

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = 1 \ . \tag{2.3}$$

The function F satisfies the generalized wave equation for disturbances in three-dimensional space,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0 \ , \tag{2.4}$$

which can also be expressed in terms of the Laplacian defined by Eq. 1.40,

$$\nabla^2 F - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0 \ . \tag{2.5}$$

This general form is applicable in numerous coordinate systems.

Retarded Time

Physically, functions such as f and F represent action at a distance retarded in time. Thus, a disturbance at point x_o at time $t = 0$ in Fig. 2.1 is experienced at a remote point x_2 at later time $t_2 = (x_2 - x_o)/c$. Introducing retarded time as

$$t' \equiv t - \frac{r}{c} \ , \tag{2.6}$$

wave functions such as f and F can be expressed simply as functions of retarded time, t'. Equation 2.5 implies action at a distance retarded in time. Whenever action at all locations occurs simultaneously, disturbances are essentially propagated with infinite speed and the second term in Eq. 2.5 is zero. Thus, when a change occurs at a boundary, instantaneous reaction everywhere can be described mathematically by an equation, called *Laplace's equation*, in which the Laplacian is zero. An equation of the form of Eq. 2.5, on the other hand, implies a propagating disturbance for which action at a distance is retarded in time.

Equation 2.5 is a general wave equation applicable to many types of propagating disturbances. Acoustic disturbances obey this wave equation when certain physical conditions are satisfied. In Chapter 5 it will be shown that bending waves in rods and plates are described by a different differential equation.

Harmonic Representation of Waves

If amplitudes of waves are small enough so that linear relationships between stress and strain apply in the medium, then several waves may be superimposed, creating new waves. More important, any arbitrary disturbance may be decomposed into a number of component periodic waves. The simplest periodic waves are, of course, sinusoids associated with simple harmonic motion. A sinusoid propagating in the x direction may be written

$$f(x - ct) = A_1 \cos\left(\omega t - \frac{\omega}{c} x\right) + A_2 \sin\left(\omega t - \frac{\omega}{c} x\right), \tag{2.7}$$

where $\omega \equiv 2\pi f$ is the angular frequency measured in radians per second. The angular frequency divided by the speed of wave propagation is essentially a spatial frequency. It is proportional to the number of wave cycles occurring in a unit distance, and is termed the *wave number*:

$$k \equiv \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}. \tag{2.8}$$

Wave number plays a role in space similar to that of angular frequency in the time domain.

The sinusoid of Eq. 2.7 may be expressed by a single cosine term:

$$f(x - ct) = \sqrt{A_1^2 + A_2^2} \cos\left(\omega t - kx - \tan^{-1}\frac{A_2}{A_1}\right). \tag{2.9}$$

Using the convention that the cosine is the real part of a complex exponential, as in Eq. 1.56, Eq. 2.9 can be written

$$f(x - ct) = RP\left(\underline{A}\, e^{i(\omega t - kx)}\right), \tag{2.10}$$

where the complex amplitude, $\underline{A}$, expresses the phase angle as well as magnitude of a rotating complex vector. In what follows, we will represent most sinusoids as complex quantities and omit RP, since "real part of" is always understood in physical equations.

The harmonic approach to wave phenomena is used almost universally. This is because it is consistent with spectral analysis, and because there are cases for which the effective wave speed, c, is a function of frequency and for which the general wave equation is therefore invalid.

Helmholtz Equation

When the solution of the wave equation is expressed by sinusoids, the equation itself takes a somewhat modified form. Since

$$\frac{\partial^2 F}{\partial t^2} = (i\omega)^2 F = -k^2 c^2 F, \tag{2.11}$$

Eq. 2.5 becomes

$$\nabla^2 F + k^2 F = 0. \tag{2.12}$$

This is the *Helmholtz equation*, and is a common form of the wave equation.

Wave Vectors

The wave number defined by Eq. 2.8 as a kind of spatial frequency is a scalar quantity, i.e., it is characterized by a number without any directional implications. However, position in space implies direction from an origin and is a vector. It is, therefore, quite useful to define a vector quantity for the spatial domain representing not only the magnitude of the wave number, but also the direction of propagation of the wave. In cartesian coordinates

$$\vec{k} \equiv \hat{i} k_x + \hat{j} k_y + \hat{k} k_z = k\left(\hat{i} n_x + \hat{j} n_y + \hat{k} n_z\right) , \tag{2.13}$$

where the various n's are the direction cosines of a unit vector normal to the plane of constant phase, as previously discussed. Since the sum of the squares of the direction cosines is unity, it follows that

$$k^2 = k_x^2 + k_y^2 + k_z^2 . \tag{2.14}$$

Using the wave vector, the general expression for a plane harmonic wave in space may be written:

$$F(x,y,z,\omega,t) = \underline{A}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} . \tag{2.15}$$

The significance of the wave vector can be illustrated in connection with the solution of the wave equation by the method of separation of variables. If the function F is expressed as the product of three spatial functions and a time function,

$$F(x,y,z,\omega,t) = X(x,\omega) \cdot Y(y,\omega) \cdot Z(z,\omega)\, e^{i\omega t} , \tag{2.16}$$

then Eq. 2.12 becomes

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + k_x^2 + k_y^2 + k_z^2 = 0 . \tag{2.17}$$

Treating this equation as three equations of the form

$$\frac{d^2 X}{dx^2} + k_x^2 X = 0 , \tag{2.18}$$

it is clear that the components of the wave vector are the constants that separate the three-dimensional wave equation into three separate equations.

Just as waves can be analyzed in terms of their spectral components in the frequency domain, they can also be analyzed in terms of their wave-number spectra. The only difference is that the analysis involves all three coordinate directions and resolves into three wave-number spectra. Fourier transforms are consequently somewhat more complex.

Radar antennas and acoustic arrays that discriminate in direction can be treated as wave-vector filters, analogous to spectral filters that respond to a band of frequencies. Wave vectors are also quite useful when dealing with propagation between two media, since the boundary can be treated as a wave-vector transformer. Radiation problems invariably concern two media and so are often analyzed by means of wave vectors.

2.2 Wave Equation for Sound in Fluids

There are a number of possible approaches to the derivation of the differential equation for propagation of acoustic disturbances in a fluid medium. The approach which is taken here treats acoustics as small-signal, non-steady (a.c.) fluid mechanics. In this approach, differential equations governing sound propagation are derived from equations of fluid mechanics by treating acoustic signals as small fluctuating disturbances. Relations used are: the continuity equation, expressing conservation of mass; the equation of motion (law of conservation of momentum), which is the statement for fluids of Newton's second law; and the stress-strain relationship, or equation of state, for a fluid.

In one approach, the continuity and momentum equations are combined prior to making any special acoustic assumptions. This approach is taken in Chapter 3 in deriving a more general equation, from which the wave equation can be derived as a special case. In the present section, the acoustic wave equation is derived by making a number of restrictive assumptions and applying them to the continuity and momentum equations before their combination. This process is illuminating, since the wave equation is strictly valid for sound only when all of the assumptions given below are satisfied, and it is important to understand the roles of the assumptions.

Assumptions

The physical assumptions used in deriving the acoustic wave equation from fluid mechanics are:

1) the fluid is isotropic, homogeneous and continuous;
2) the fluid cannot withstand static shear stresses in the manner of a solid;
3) viscous stresses are negligible;
4) there is no conduction or radiation of heat;
5) any chemical, electromagnetic or other external forces experienced by the fluid are negligible;
6) there are no local sources of sound;
7) the only steady motion of the medium is a uniform constant translation;
8) the stress-strain relationship is linear;
9) the relative compression of the medium is very small ($\Delta\rho << \rho_o$);
10) particle motions associated with sound waves are irrotational; and
11) spatial variations of the ambient ~~pressure,~~ density and temperature are relatively very small.

These assumptions are required in order to derive a simple equation. To the extent that they are not valid, additional terms occur in the final equation. Most of these additional terms may be treated as source terms, but some of them invalidate the wave solution.

The basic *acoustic assumption* is that physical quantities in fluid mechanics can be expressed as sums of steady-state, time-independent values plus fluctuating acoustic values. Thus, the static pressure is expressed by

$$p(x,y,z,t) = p_o(x,y,z) + p'(x,y,z,t) \; , \tag{2.19}$$

where

$$p_o = \frac{1}{T} \int_o^T p dt \tag{2.20}$$

is the ambient value that exists when sound is absent and p' is the acoustic component, the long-time average of which is zero. Similar expressions apply to density and to the components of velocity. However, the seventh assumption listed above, that of constant translational velocity, implies that the equations can be written for a coordinate system moving with the fluid, for which $\vec{v}_o = 0$.

Equation of State

The equation of state of a substance is a relationship between static pressure, density and temperature. At a fixed temperature, pressure may be expressed as a power expansion of density, as

$$p = p_o + a(\rho - \rho_o) + b(\rho - \rho_o)^2 + \ldots , \tag{2.21}$$

where the coefficients a and b, as well as p_o and ρ_o, are functions of temperature. From the eighth and ninth assumptions, it follows that in linear acoustics higher order terms are negligible and that the acoustic pressure, p', can be related to the acoustic component of the density, ρ', by

$$p' = p - p_o \doteq a(\rho - \rho_o) = a\rho' , \tag{2.22}$$

which is the equation of state for an acoustic disturbance. From the eleventh assumption, coefficient a is assumed to be constant or, if varying, to be a slowly varying function of position.

Equation of Continuity

The continuity equation of fluid mechanics expresses conservation of mass. It can be derived by either of two approaches. In one approach, named after Lagrange, attention is focused on a particular mass of fluid as it moves through space. Continuity simply states that this mass must be constant:

$$\frac{Dm}{Dt} = \frac{D}{Dt} \int_V \rho dV = 0 . \tag{2.23}$$

The special type of derivative represented by D/Dt applies to a particle as it moves. It is called a *material derivative*, or *substantial*, and has all the mathematical attributes of a total derivative with respect to time. Since the particle moves through space, the material derivative can be expressed in terms of particle velocity, $\vec{v}$, and local partial derivatives by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \cdot \frac{\partial}{\partial y} + \frac{\partial z}{\partial t} \cdot \frac{\partial}{\partial z} \tag{2.24a}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla) = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} . \tag{2.24b}$$

Material derivatives can be applied to vector as well as scalar quantities.

As a particle moves through space, its density may change. Since its mass is constant, the volume it occupies must also change. The *transport theorem*, originally derived by Euler, relates the material derivative of an element of volume to the divergence of the velocity field:

$$\frac{D}{Dt} \int_V dV = \int_V div\, \vec{v}\, dV . \tag{2.25}$$

Using this relationship, Eq. 2.23 can be expanded:

$$\frac{Dm}{Dt} = \int_V \left(\frac{D\rho}{Dt} + \rho\, div\, \vec{v} \right) dV = 0 . \tag{2.26}$$

Since the volume is finite, it follows that the expression within the parentheses must be zero,

$$\frac{D\rho}{Dt} + \rho \left(\nabla \cdot \vec{v} \right) = 0 , \tag{2.27}$$

which is the Lagrangian form of the equation of continuity. Expanding the material derivative of the density by Eq. 2.24b yields a second form,

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \left(\nabla \cdot \vec{v} \right) = 0 , \tag{2.28}$$

which is the one used in deriving a continuity equation for acoustic disturbances.

The second method of deriving the continuity equation of fluid mechanics is named for Euler. In this approach, attention is focused on a fixed volume and the time-rate-of-change of mass within this volume is equated to the flux of mass into the volume through its surfaces:

$$\frac{\partial}{\partial t} \int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t} dV = - \int_S \rho \vec{v} \cdot \vec{dS} . \tag{2.29}$$

The surface integral can be expressed as a volume integral by invoking Gauss' divergence theorem,

$$\int_S \vec{A} \cdot \vec{dS} = \int_V div\, \vec{A}\, dV , \tag{2.30}$$

whereupon Eq. 2.29 becomes

$$\int_V \left(\frac{\partial \rho}{\partial t} + div \left(\rho \vec{v} \right) \right) dV = 0 \; , \tag{2.31}$$

and it follows that

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{v}) = \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0 \; , \tag{2.32}$$

which is the Eulerian form of the continuity equation. When the divergence term is expanded, this result is identical to that of Eq. 2.28.

The acoustic form of the equation of continuity can be derived from Eq. 2.28 by expressing each physical variable as the sum of a time-independent average value and a fluctuating component, and taking a coordinate system moving with the fluid:

$$\frac{\partial \rho'}{\partial t} + \vec{v}' \cdot \nabla \left(\rho_o + \rho' \right) + \left(\rho_o + \rho' \right) \left(\nabla \cdot \vec{v}' \right) = 0 \; . \tag{2.33}$$

The gradient of ρ_o is negligible by the eleventh assumption, and ρ is negligible relative to ρ_o in the third term by the ninth assumption, leaving

$$\frac{\partial \rho'}{\partial t} + \vec{v}' \cdot \nabla \rho' + \rho_o \left(\nabla \cdot \vec{v}' \right) = 0 \; . \tag{2.34}$$

The first and last terms exhibit linear dependencies on fluctuating quantities, while the middle term is quadratic. In the limit, for very small acoustic fluctuations, this term must become negligible relative to the other two. The final form of the linear acoustic continuity equation in a region free of acoustic sources is

$$\frac{\partial \rho'}{\partial t} + \rho_o \left(\nabla \cdot \vec{v}' \right) = \frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i'}{\partial x_i} = 0 \; . \tag{2.35}$$

Equation of Motion

The equation of motion for a fluid may be formulated directly from Newton's second law by equating the rate-of-change of momentum of a fluid particle to the sum of the forces acting on it. Forces which are considered in fluid mechanics include gravity, the gradient of pressure, viscous stresses and other unspecified external forces. However, in an acoustic derivation, by the first six assumptions, it is only necessary to consider forces associated with gravity and with the gradient of the pressure.

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The gravitational force experienced by a particle is given by

$$\vec{F}_g = m\vec{g} = \int_V \rho\vec{g}\, dV = - \int_V \rho g \nabla z\, dV \ . \tag{2.36}$$

where the minus sign shows that gravity is a downward force, the z-axis being positive upward.

Pressure applies force normal to surfaces of a volume,

$$\vec{F}_p = - \int_S p\, d\vec{S} \ , \tag{2.37}$$

where the minus sign arises from the inward direction of pressure-generated forces and use of the outward normal in defining a vector surface element. Surface integrals can be transformed to volume integrals by use of Gauss' gradient theorem,

$$\int_S A\, d\vec{S} = \int_V \text{grad}\, A\, dV \ , \tag{2.38}$$

and the net force caused by the gradient of the pressure becomes

$$\vec{F}_p = - \int_V \nabla p\, dV \ . \tag{2.39}$$

The momentum of a particle is the volume integral of the product of its density and velocity. Taking the material derivative, expanding it by means of Eq. 2.24 and using the transport theorem, Eq. 2.25, one obtains

$$\frac{D\vec{M}}{Dt} = \frac{D}{Dt}\int_V (\rho\vec{v})\,dV = \int_V \left(\frac{D(\rho\vec{v})}{Dt} + (\rho\vec{v})\,(\nabla\cdot\vec{v})\right)dV$$

$$= \int_V \left(\rho\frac{D\vec{v}}{Dt} + \vec{v}\frac{D\rho}{Dt} + (\rho\vec{v})\;(\nabla\cdot\vec{v})\right)dV \ . \tag{2.40}$$

From the continuity equation as given by Eq. 2.27, the second and third terms are seen to add to zero leaving only the first term. Equating the rate-of-change of momentum to the sum of the forces,

$$\frac{d\vec{M}}{dt} = \int_V \rho\frac{D\vec{v}}{Dt}\,dV = \vec{F}_g + \vec{F}_p = - \int_V \left(\rho g \nabla z + \nabla p\right) dV \ . \tag{2.41}$$

it follows that

$$\rho \frac{D\vec{v}}{Dt} = - \rho g \nabla z - \nabla p \,, \tag{2.42}$$

which is the Lagrangian form of the momentum equation in an ideal inviscid fluid. Expanding the material derivative by Eq. 2.24 yields

$$\rho \frac{\partial \vec{v}}{\partial t} = - \left(\rho g \nabla z + \nabla p + \rho (\vec{v} \cdot \nabla) \vec{v} \right), \tag{2.43}$$

which is a form useful for acoustic derivations.

As for continuity, the acoustic form of the momentum equation is obtained by replacing each physical variable by the sum of its steady and fluctuating components, and by taking a coordinate system moving with the fluid, so that $\vec{v}_o = 0$. When this is done, Eq. 2.43 becomes

$$(\rho_o + \rho') \frac{\partial v'}{\partial t} = - (\rho_o + \rho') g \nabla z - \nabla (p_o + p') - (\rho_o + \rho')(\vec{v}' \cdot \nabla) \vec{v}' \,. \tag{2.44}$$

Making the ninth assumption and retaining only linear terms,

$$\rho_o \frac{\partial \vec{v}'}{\partial t} = - \left(\rho_o g \nabla z + \nabla p_o + \nabla p' \right). \tag{2.45}$$

Since this equation is also valid in the absence of sound, the gradient of the ambient pressure cancels the gravitational term,

$$\nabla p_o = - \rho_o g \nabla z \,, \tag{2.46}$$

leaving

$$\rho_o \frac{\partial \vec{v}'}{\partial t} = - \nabla p' \,, \tag{2.47}$$

which is the acoustic conservation of momentum equation for an ideal fluid medium free of external sources.

It is in the next to the last step that derivations for liquids and gases may differ. The derivation given here is valid for all non-viscous fluids. However, when deriving the acoustic momentum equation for gases, it is common practice to ignore gravitational forces and to assume that the gradient of the ambient pressure is of second order. While valid for gases, this procedure is not valid in liquids. It is also common practice to assume that pressure fluctuations are small, an assumption that is often not valid in liquids and which is not made in the present derivation.

Acoustic Wave Equation

Equations 2.35, 2.47 and 2.22 for continuity, momentum and state can be combined to derive a second-order differential equation for acoustic quantities. Taking the partial derivative of

Eq. 2.35 with respect to time,

$$\frac{\partial^2 \rho'}{\partial t^2} + \frac{\partial}{\partial t}\left(\rho_o \nabla \cdot \vec{v}'\right) = \frac{\partial^2 \rho'}{\partial t^2} + \rho_o\left(\nabla \cdot \frac{\partial \vec{v}'}{\partial t}\right) = 0 \ . \tag{2.48}$$

since the order of differentiation is immaterial. Taking the divergence of the momentum equation, Eq. 2.47, yields

$$\nabla \cdot \rho_o \frac{\partial \vec{v}'}{\partial t} \doteq \rho_o\left(\nabla \cdot \frac{\partial \vec{v}'}{\partial t}\right) = -\nabla^2 p' \ , \tag{2.49}$$

where a term involving grad ρ_o has been assumed to be of second order, in accordance with the eleventh assumption. Substituting Eq. 2.49 for the second term in Eq. 2.48,

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = 0 \ . \tag{2.50}$$

The equation of state can now be used to eliminate either acoustic density or pressure. The results are similar. Thus, using Eq. 2.22 for p' and assuming that spatial derivatives of a are negligible,

$$\frac{\partial^2 \rho'}{\partial t^2} - a\nabla^2 \rho' = 0 \ . \tag{2.51}$$

Dividing both terms by $-a$ yields a more common form,

$$\nabla^2 \rho' - \frac{1}{a}\frac{\partial^2 \rho'}{\partial t^2} = 0 \ , \tag{2.52}$$

which we recognize as a wave equation, since it is similar to Eq. 2.5.

Comparing Eq. 2.52 to Eq. 2.5, it is apparent that the constant a in Eq. 2.52 equals the square of the wave speed, c. Since the speed of sound is a property of the medium, it is represented by c_o. From Eq. 2.22, c_o is a function of the compressibility of the medium,

$$c_o = \sqrt{a} = \sqrt{\frac{p'}{\rho'}} = \sqrt{\frac{dp}{d\rho}} \doteq \sqrt{\frac{B}{\rho_o}} \ , \tag{2.53}$$

where B is the bulk modulus and expresses resistance to compression.

It is of interest to note that as the medium becomes more and more incompressible the speed of sound approaches infinity and the wave equation approaches Laplace's equation. Solutions of Laplace's equation are uniquely determined by boundary conditions: any changes in values at a boundary are felt immediately throughout the entire medium. In acoustics, changes at a boundary are experienced throughout a medium at later times. Thus, the fundamental distinction between acoustics and hydrodynamics is the delay of responses at a distance, delay being implied by the finiteness of the second term of the wave equation. Stated another way, Laplace's equation is valid when the largest dimensions involved in a problem are small compared to an acoustic wavelength. When physical dimensions become comparable to a wavelength, then compressibility can no longer be ignored and the wave equation must be used.

Velocity Potential

Wave equations of the same form as Eq. 2.52 can be derived for acoustic pressure, p', and for each component of particle velocity, $\vec{v}'$. Since by the tenth assumption particle motions in sound waves are irrotational, it is also possible to define a scalar velocity potential by

$$\vec{v}' \equiv - \, grad \, \phi \; , \tag{2.54}$$

and to relate density and pressure fluctuations to this acoustic potential. Substituting for the velocity term in the acoustic momentum equation, Eq. 2.47 becomes

$$\rho_o \nabla \left(\frac{\partial \phi}{\partial t} \right) = \nabla p' \; . \tag{2.55}$$

from which it follows that

$$p' = \rho_o \frac{\partial \phi}{\partial t} \; , \tag{2.56}$$

since by the very definition of a fluctuating component all constants of integration must be zero. From the equation of state, Eq. 2.53, the acoustic density is given by

$$\rho' = \frac{p'}{c_o^2} = \frac{\rho_o}{c_o^2} \frac{\partial \phi}{\partial t} \; . \tag{2.57}$$

Acoustic potential also satisfies the wave equation, as can be shown by substituting Eq. 2.57 for ρ' and Eq. 2.54 for $\vec{v}'$ into the continuity equation, Eq. 2.35:

$$\frac{\partial}{\partial t} \left(\frac{\rho_o}{c_o^2} \frac{\partial \phi}{\partial t} \right) - \rho_o \nabla \cdot \nabla \phi = 0 \; . \tag{2.58}$$

Therefore,

$$\nabla^2 \phi - \frac{1}{c_o^2} \ddot{\phi} = 0 \; , \tag{2.59}$$

where each dot over ϕ represents a differentiation with respect to time.

Harmonic Solutions

The assumption of linearity made in deriving the acoustic wave equation makes it possible to treat any arbitrary disturbance as the sum of sinusoidal components, each of the form

$$\phi(\omega) = RP(\underline{\phi}) = RP(\Phi e^{i\omega t}) \; , \tag{2.60}$$

where the complex amplitude, Φ, is a function of position in space as well as frequency and is found by solving the Helmholtz equation, Eq. 2.12:

$$\nabla^2 \underline{\Phi} + k^2 \underline{\Phi} = 0 \; . \qquad (2.61)$$

Once the potential is known, the acoustic pressure and density can readily be found from

$$\underline{p}' = \rho_o \dot{\underline{\phi}} = i\omega\rho_o \underline{\phi} = ik\rho_o c_o \underline{\phi} \qquad (2.62)$$

and

$$\underline{\rho}' = \frac{\rho_o}{c_o^2} \dot{\underline{\phi}} = i\omega \frac{\rho_o}{c_o^2} \underline{\phi} = ik \frac{\rho_o}{c_o^2} \underline{\phi} \; . \qquad (2.63)$$

However, the particle velocity, being a function of the gradient of the potential, depends upon the particular spatial solution. Of the many possible solutions, the two most useful are for plane and spherical waves. Since most other waves can be treated as superpositions of either of these fundamental types, we will limit our discussions of solutions of Eq. 2.61 to plane and spherical waves.

2.3 Plane Sound Waves

Acoustic Potential

The scalar potential describing a plane harmonic sound wave can be written

$$\underline{\phi} = \underline{\Phi}\, e^{i\omega t} = \underline{A}\, e^{i(\omega t - \vec{k}\cdot\vec{r})} = \underline{A}\, e^{i(\omega t - k_x x - k_y y - k_z z)} \; , \qquad (2.64)$$

which can be verified either by carrying out the indicated differentiations and substituting the results into the Helmholtz equation or by direct comparison with the expressions given in Eqs. 2.15 and 2.16. The amplitude $\underline{A}$ is constant as the wave progresses.

Particle Velocity

The particle velocity, $\vec{\underline{v}}'$, is related through Eq. 2.54 to the gradient of the scalar potential. In cartesian coordinates, for a plane wave

$$\vec{\underline{v}}' = -\nabla \underline{\phi} = -\left(\hat{\imath}\frac{\partial \underline{\phi}}{\partial x} + \hat{\jmath}\frac{\partial \underline{\phi}}{\partial y} + \hat{k}\frac{\partial \underline{\phi}}{\partial z} \right) = i\left(\hat{\imath} k_x \underline{\phi} + \hat{\jmath} k_y \underline{\phi} + \hat{k} k_z \underline{\phi} \right) = i\vec{k}\underline{\phi} \; . \qquad (2.65)$$

which is a vector having the same direction as the wave vector and having an instantaneous value given by the real part of $ik\underline{\phi}$. Since the direction of propagation is usually obvious from geometrical considerations, it is common to deal with the acoustic particle speed and to write

$$\underline{v}' = ik\underline{\phi} \; . \qquad (2.66)$$

Comparison of this expression with Eqs. 2.56 and 2.57 for acoustic pressure and density shows that for a plane wave the particle speed is in phase with both pressure and density fluctuations. This differs from hydrodynamics, for which velocity and pressure changes are often 180° out of phase.

The ratio of the particle velocity to the sound speed of the medium is essentially the *Mach* *number, M'*, of the acoustic particle motion. One would expect that second-order effects might become important if this Mach number approached unity. Actually, for plane waves the expression for acoustic Mach number is identical to that for the ratio of the fluctuating density to its steady-state value, as given by Eq. 2.63:

$$M' \equiv \frac{\underline{v}'}{c_o} = i\frac{k}{c_o}\underline{\phi} = \frac{\underline{\rho}'}{\rho_o} . \tag{2.67}$$

Since, by the ninth assumption, the density ratio is assumed to be very small when deriving the wave equation, it follows that acoustic Mach numbers are also small.

Specific Acoustic Impedance

The concept of mechanical impedance is often used when dealing with mechanical systems to express the ratio of a force to a velocity. A similar concept is used in acoustics when dealing with forces experienced by a radiating surface. Since by Eqs. 2.62 and 2.66 the acoustic pressure and particle speed in a wave are both proportional to acoustic potential, their ratio is a constant which is called the *specific acoustic impedance*. For a plane wave,

$$\underline{z}_a \equiv \frac{\underline{p}'}{\underline{v}'} = \frac{i\omega\rho_o\underline{\phi}}{ik\underline{\phi}} = \frac{\omega\rho_o}{k} = \rho_o c_o . \tag{2.68}$$

The quantity $\rho_o c_o$ is a property of the medium. It is called the *characteristic impedance* of the medium and is measured in units called *Rayls*, named after Lord Rayleigh. The value for water is close to 1.5×10^6 MKS Rayls, while the corresponding value for standard air is only 415. The difference is indicative of the relative compressibilities of the two media.

Acoustic Intensity

The fact that there is both particle motion and medium compression associated with a sound wave implies that there are both kinetic and potential energies in sound waves. However, it is possible in a standing wave to have kinetic and potential energies without any net flow of energy from one place to another. We are really more interested in the transfer of power by an acoustic disturbance than in the energy *per se*. The quantity which measures the transfer of acoustic power across a unit area is called the *acoustic intensity*, *I*. Intensity is the time-average power flow per unit area normal to the direction of travel of the wave and is given by

$$I \equiv \frac{1}{T}\int_0^T p'(t)v'(t)\,dt = \overline{p'(t)v'(t)} , \tag{2.69}$$

where $p'(t)$ and $v'(t)$ are instantaneous values of the pressure and the particle speed.

In a progressing plane wave, pressure and particle speed are in phase with each other. The speed can be expressed in terms of pressure by Eq. 2.68 and the intensity is given by

$$I = \frac{1}{T}\int_0^T p'(t) \cdot \frac{p'(t)}{\rho_o c_o}\,dt = \frac{\overline{p'^2}}{\rho_o c_o} . \tag{2.70}$$

One could also use Eq. 2.68 to substitute for the acoustic pressure, deriving an expression for intensity in terms of the rms particle speed,

$$I = \frac{1}{T}\int_{0}^{T} \rho_o c_o \mathrm{v}'(t) \cdot \mathrm{v}'(t)\, dt = \rho_o c_o \overline{\mathrm{v}'^2} \ . \tag{2.71}$$

It also can be shown that the intensity of a plane wave is simply the product of the rms pressure and rms particle speed.

In a standing wave, pressures of the two waves are cumulative, but particle velocities cancel on average. Hence, Eq. 2.69 yields zero intensity for an ideal plane standing wave.

2.4 Spherical Waves

Acoustic Potential

There are many instances in underwater acoustics when a source can be treated as a small pulsating spherical surface radiating sound in all directions. For this situation, the Laplacian is written in spherical coordinates for spherical symmetry by Eq. 1.44, and the Helmholtz equation becomes

$$\nabla^2 \underline{\phi} + k^2 \underline{\phi} = \frac{1}{r}\frac{\partial^2 (r\underline{\phi})}{\partial r^2} + k^2 \underline{\phi} = 0 \ . \tag{2.72}$$

Multiplying through by r, one finds

$$\frac{\partial^2 (r\underline{\phi})}{\partial r^2} + k^2 (r\underline{\phi}) = 0 \ , \tag{2.73}$$

which is of the same form as a one-dimensional Helmholtz equation, with $(r\underline{\phi})$ replacing $\underline{\phi}$ and r replacing x. It follows that the solution is of the form

$$\underline{\phi} = \frac{A}{r} e^{i(\omega t - kr)} \ , \tag{2.74}$$

showing that the magnitude of the potential decreases inversely with increasing distance from a source of spherical waves.

Since Eqs. 2.56 and 2.57 for acoustic pressure and density involve only time derivatives of the acoustic potential, these quantities bear the same relation to the potential as for plane waves, Eqs. 2.62 and 2.63. Their amplitudes therefore vary inversely with distance, in the same manner as the potential. Thus

$$P(r) = \frac{1}{r} P(1) \ . \tag{2.75}$$

Particle Velocity

Equation 2.54 for particle velocity depends on spatial derivatives of the acoustic potential, and so depends on the shape of the wave front. For spherical waves,

$$\underline{\vec{v}}' = -\nabla \underline{\phi} = -\hat{r}\frac{\partial \underline{\phi}}{\partial r} = \hat{r}\left(\frac{\underline{\phi}}{r} + ik\underline{\phi}\right). \qquad (2.76)$$

It follows that the particle speed can be expressed in either of two forms,

$$\underline{v}' = \frac{\underline{\phi}}{r}(1 + ikr) = ik\underline{\phi}\left(1 - \frac{i}{kr}\right), \qquad (2.77)$$

from which it is clear that basically there are two distinct regimes. Close to a radiating source, in the *near field* given by $kr << 1$, the particle speed is dominated by the term that is in phase with the potential. Far from a source, in the *far field*, the particle speed is dominated by the term that is out of phase with the potential but in phase with the acoustic density and pressure. In the near field, particle speed varies inversely with the square of distance, since it drops off faster than the potential. In the far field, its dependence on distance is the same as that of the potential.

Specific Acoustic Impedance

Since the particle speed has both near- and far-field terms, the specific acoustic impedance is also a function of relative distance:

$$\underline{z}_a \equiv \frac{\underline{p}'}{\underline{v}'} = \frac{i\omega\rho_o\underline{\phi}}{ik\underline{\phi}\,(1 - (i/kr))} = \rho_o c_o \frac{(kr)^2 + ikr}{1 + (kr)^2}. \qquad (2.78)$$

In the near field, $kr < 1$, the impedance is dominantly reactive, while in the far field it is basically resistive and for large kr approaches the plane-wave value.

The resistive and reactive components form a complex number which can be expressed in exponential form,

$$\underline{z}_a = \rho_o c_o \frac{kr}{\sqrt{1 + (kr)^2}} \cdot \frac{kr + i}{\sqrt{1 + (kr)^2}}$$

$$= \rho_o c_o \cos\theta\,(\cos\theta + i\sin\theta) = \rho_o c_o \cos\theta\, e^{i\theta}, \qquad (2.79)$$

where

$$\theta = \tan^{-1}\frac{1}{kr} = \frac{\pi}{2} - \tan^{-1} kr. \qquad (2.80)$$

In the near field, $kr << 1$ and θ approaches $\pi/2$, from which it follows that

$$\underline{z}_a \doteq ikr\rho_o c_o \quad (kr << 1). \qquad (2.81)$$

In the far field, on the other hand, θ approaches 0 and Eq. 2.79 reduces to Eq. 2.68 for the impedance of a plane wave.

Acoustic Intensity

Equation 2.69 for acoustic intensity involves the product of instantaneous values of the acoustic pressure and particle speed. In complex notation, intensity is the time-average value of the product of the components of pressure and particle speed that are in phase with each other. Combining Eqs. 2.62 and 2.77, the particle speed of a spherical wave is related to its acoustic pressure by

$$\underline{v}' = \frac{\underline{p}'}{\rho_o c_o}\left(1 - \frac{i}{kr}\right). \tag{2.82}$$

The component in phase with the pressure is clearly the first term, which is independent of kr and is therefore the same in the near field as in the far field. From Eqs. 2.69 and 2.82, the intensity of a spherical wave is

$$I = \frac{1}{T}\int_0^T p'(t) \cdot \frac{p'(t)}{\rho_o c_o}\, dt = \frac{\overline{p'^2}}{\rho_o c_o}. \tag{2.83}$$

This is the same as Eq. 2.70 for plane waves. If, instead of substituting for the particle speed, one were to substitute for the pressure, then from Eq. 2.79

$$\underline{p}' = \underline{v}' \frac{\rho_o c_o (kr)^2}{1 + (kr)^2}\left(1 + \frac{i}{kr}\right) = \underline{v}' \rho_o c_o \cos^2\theta \left(1 + \frac{i}{kr}\right), \tag{2.84}$$

and the intensity is

$$I = \frac{1}{T}\int_0^T v'(t) \cdot v'(t) \rho_o c_o \cos^2\theta\, dt = \rho_o c_o \overline{v'^2} \cos^2\theta , \tag{2.85}$$

which differs from the plane-wave expression by $\cos^2\theta$.

Because of the general validity of Eq. 2.83, pressure-sensitive instruments are better indicators of acoustic intensity than are velocity-sensitive ones. In water, because of the high impedance of the medium, almost all measurements are made with pressure-sensitive transducers and Eq. 2.83 is used to estimate the intensity. This procedure is valid when there is only one source and sound waves are progressing in only one direction. If waves emanate from more than one source, the particle velocities will have different directions and may even cancel. For example, a standing wave is formed when two equal waves are progressing in opposite directions. Such a wave carries no energy and has zero intensity, even though the rms pressure is finite.

Ideal Transmission Loss

Since the intensity of a spherical wave is proportional to the square of the pressure, and pressure according to Eq. 2.75 is inversely proportional to distance, it follows that the intensity from a simple source decays as the square of the distance. For this reason, ideal spherical sound propagation is often called *inverse-square spreading*.

As discussed in Chapter 1, a logarithmic quantity, transmission loss, is often used to express changes of acoustic intensity and pressure with distance. Substituting Eq. 2.75 into Eq. 1.20, which defines transmission loss, it follows that in an ideal, lossless medium transmission loss is given simply by

$$TL_i \equiv 20 \log \frac{p(1)}{p(r)} = 20 \log r \ , \tag{2.86}$$

and that in such an ideal medium sound pressure levels decrease by 6 dB for every doubling of distance.

Equation 2.86 for the transmission loss for spherical waves in an ideal medium is a direct consequence of the wave equation and of the assumptions made in deriving it. To the extent that these assumptions are not met, actual transmission loss will differ from ideal spherical spreading. *Transmission anomaly*, *TA*, is defined as the difference of the actual transmission loss from that predicted by spherical spreading:

$$TA \equiv TL - TL_i = TL - 20 \log r \ . \tag{2.87}$$

Anomaly is thus the dB measure of the cumulative effects of all of the ways in which the actual medium differs from the ideal medium assumed in the derivation of the wave equation. It is positive when the measured transmission loss exceeds the ideal value.

Acoustic Power

The total power radiated by a source can be obtained by integrating the intensity over a spherical surface:

$$W_{ac} = \int_S I \, dS \doteq \int_S \frac{\overline{p'^2}}{\rho_o c_o} \, dS \ . \tag{2.88}$$

In the most general case, radiation does not occur uniformly and both I and p' are functions of angle as well as distance. In the case of an omnidirectional source,

$$W_{ac} = \frac{\overline{p'^2}}{\rho_o c_o} \cdot 4\pi r^2 \doteq 4\pi \frac{\overline{p'(1)^2}}{\rho_o c_o} \ , \tag{2.89}$$

showing that a consequence of the inverse-square law is constancy of acoustic power as a function of distance from a source.

It is because of the dependence of the acoustic power on $\rho_o c_o$ indicated by Eq. 2.88 that the acoustic power for a given acoustic pressure is so much lower in water than in air. Since the product of the density and speed of sound of water is approximately 3600 times as great as that of air, the same acoustic power produces acoustic pressure 60 times larger in water than in air.

Damped Sound Waves

The wave equation was derived under the assumption that viscosity, heat conduction and other dissipative phenomena are negligible. In actual sound transmission in the ocean these effects, while very small, are not zero. They may be taken into account by a perturbation approach in which the wave speed is treated as a complex quantity for which the dissipative out-of-phase component is very small compared to the real part:

$$\underline{c} = c(1 + i\eta) \qquad \eta << 1 . \tag{2.90}$$

The wave number is then also complex, and the expression for a spherical wave becomes

$$\underline{\phi} = \frac{A}{r} e^{i(\omega t - \underline{k} r)} = \frac{A}{r} e^{-k\eta r} e^{i(\omega t - kr)} . \tag{2.91}$$

This expression represents a spherical wave whose amplitude decays at a slightly greater rate than inversely with distance.

The effect of the real exponential term in Eq. 2.91 is to increase the transmission loss relative to the ideal value given by Eq. 2.86. Thus

$$TL = 20 \log \frac{re^{k\eta r}}{e^{k\eta}} \doteq 20 \log r + 20 \log e^{k\eta r} , \tag{2.92}$$

since $k\eta << 1$. The dissipative transmission anomaly is therefore

$$TA_{dis} = 20 \log e^{k\eta r} = (20)(0.4343)k\eta r = 8.686k\eta r . \tag{2.93}$$

It is usual to write $TA = \alpha r$ where

$$\alpha = 8.686k\eta \tag{2.94}$$

is the absorption coefficient in dB/m. The transmission loss in a slightly dissipative medium is thus

$$TL = TL_i + TA_{dis} = 20 \log r + \alpha r . \tag{2.95}$$

For this perturbation approach to the damping of sound waves to be valid, it is necessary that η be a very small quantity indeed. Measurements in sea water show that η increases with frequency, and that for frequencies under one megahertz it is smaller than 2×10^{-5}. Equation 2.91 is thus an excellent approximation for the effect of damping on spherical waves in water over the entire frequency range generally exploited by underwater sound devices and systems.

Spherical Waves from Plane Waves

In most problems spherical symmetry, with its inverse radial dependence of pressure and density, is easy to handle analytically. However, for some cases, such as the interaction of spherical waves with plane surfaces, spherical symmetry is a handicap. For such problems, it is useful to treat spherical waves as the superposition of an infinite number of plane waves. Brekhovskikh

(1960) has shown that

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi} \iint_{-\infty}^{\infty} \frac{e^{i\left(k_x x + k_y y + k_z z\right)}}{k_z} \, dk_x dk_y \ . \tag{2.96}$$

where the integration is in wave-vector space. With this transformation spherical waves can be replaced with plane waves when that is desirable.

2.5 Transmission at Media Interfaces

Snell's Law

In many instances sound waves created in one fluid medium are received in, or reflected by, a second medium. Problems involving planar boundaries are best treated by considering plane waves. As we have just noted, spherical waves can be decomposed into plane waves. If the second medium is a fluid, then the boundary is incapable of sustaining a stress, and the components of wave velocity parallel to the boundary must be the same in both media. It follows that the components of the wave vector parallel to a boundary surface are unchanged in crossing between the two media. This recognition of the constancy of the components of the wave vectors parallel to a boundary between two fluids leads directly to a derivation of *Snell's Law*.

Figure 2.2 represents the geometric picture of incident, reflected and transmitted rays at a

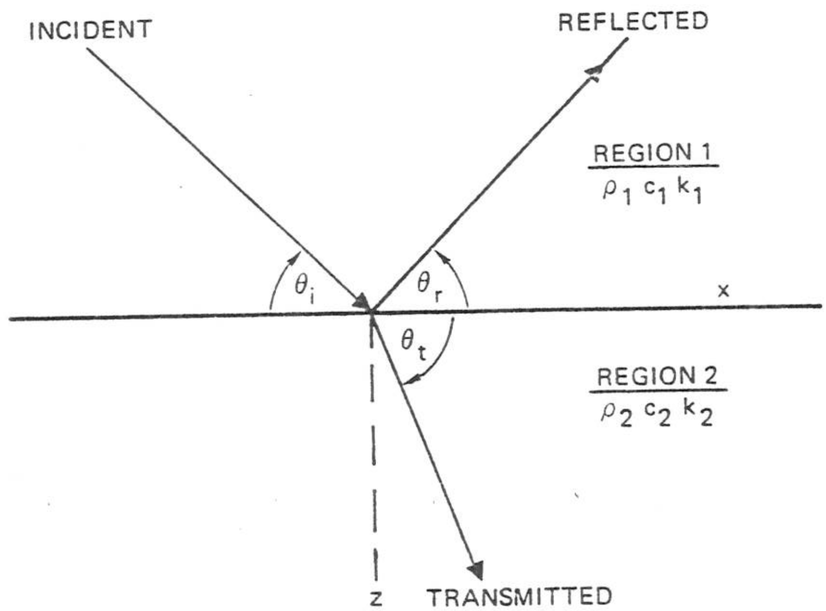

Fig. 2.2. Transmission at a Boundary between Two Fluid Media

boundary between two fluids. The parallel components of the three wave vectors are

$$k_{i_x} = k_1 \cos \theta_i \ . \tag{2.97}$$

$$k_{r_x} = k_1 \cos \theta_r \tag{2.98}$$

and

$$k_{t_x} = k_2 \cos \theta_t \ . \tag{2.99}$$

where $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$, and angles are measured relative to the plane rather than to a normal as is more common in airborne acoustics. Since all three parallel components are equal, it follows that

$$\theta_r = \theta_i \tag{2.100}$$

and

$$\cos \theta_t = \frac{c_2}{c_1} \cos \theta_i \ . \tag{2.101}$$

This last relationship is Snell's law. It can be generalized to cases where sound speed is continuous, leading to the general statement that for any ray in a refractive medium the component of the wave vector parallel to the isovelocity surface is constant.

Reflection from a Plane Boundary

Equations 2.100 and 2.101 relate the angles of reflected and transmitted rays to the incident ray angle. Relationships between the amplitudes can be derived by recognizing that both the pressure and normal particle velocity must be continuous. Continuity of pressure requires that

$$P_i + P_r = P_t \ . \tag{2.102}$$

where the various P's are amplitudes of the respective acoustic pressures. Continuity of the normal particle velocity leads to

$$P_i - P_r = \beta P_t \ . \tag{2.103}$$

where

$$\beta \equiv \frac{\rho_1 c_1}{\rho_2 c_2} \frac{\sin \theta_t}{\sin \theta_i} = \frac{\rho_1}{\rho_2} \frac{\tan \theta_t}{\tan \theta_i} \ . \tag{2.104}$$

Simultaneous solution of Eqs. 2.102 and 2.103 leads to a relation for the ratio of reflected pressure to incident pressure,

$$\alpha_r \equiv \frac{P_r}{P_i} = \frac{1 - \beta}{1 + \beta} \ . \tag{2.105}$$

which is valid if sin θ_t exists. When Eq. 101 is solved for sin θ_t, it is found that

$$sin\ \theta_t = \sqrt{1 - \left(\frac{c_2}{c_1}\right)^2 cos^2\ \theta_i}\ . \qquad (2.106)$$

and that energy is transmitted into a medium with higher speed of sound only if the incident angle is greater than a *critical angle* defined by

$$\theta_c = cos^{-1}\left(\frac{c_1}{c_2}\right)\ . \qquad (2.107)$$

When grazing angles of incidence are less than the critical angle, all of the energy is reflected.

Transmission through a Plane Boundary

Simultaneous solution of Eqs. 2.102 and 2.103 for P_t gives

$$\alpha_t \equiv \frac{P_t}{P_i} = \frac{2}{1 + \beta}\ , \qquad (2.108)$$

provided, of course, that θ_i is greater than θ_c. However, this equation for the pressure transmission ratio does not tell the whole story. Because the specific acoustic impedances of the two media differ, the relative sound power transmitted is not simply the square of the transmitted pressure ratio. Transmitted power can be calculated by subtracting reflected power from incident power, from which it follows that

$$\frac{W_t}{W_i} = 1 - \frac{W_r}{W_i} = 1 - \alpha_r^2 = \frac{4\beta}{(1 + \beta)^2}\ . \qquad (2.109)$$

In treating transmission loss between two media, distinction needs to be made between pressure and power transmission losses. From Eq. 2.108 the pressure transmission loss is

$$TL \equiv 20\ log\frac{P_i}{P_t} = 20\ log\frac{1 + \beta}{2}\ , \qquad (2.110)$$

while the power loss is

$$PTL \equiv 10\ log\frac{W_i}{W_t} = 10\ log\frac{(1 + \beta)^2}{4\beta} = TL - 10\ log\ \beta\ . \qquad (2.111)$$

Transmission from Air to Water

Sometimes when calculating noise in water it is necessary to estimate contributions of sources in air. For example, one may wish to calculate the sound in water caused by aircraft flying overhead. The speed of sound in water is 4.35 times that of air. From Eq. 2.107, it follows that sound is transmitted from air into water only if the incident angle exceeds about 75°. The ratio of the specific acoustic impedances of the two media is 3600, from which

$$\beta \doteq 2.8 \times 10^{-4} \sin \theta_t \; . \qquad (2.112)$$

The most efficient transmission of sound from air into water occurs near normal incidence, for which

$$\alpha_t = \frac{2}{1 + \beta} \doteq 2 \qquad (2.113)$$

and

$$\frac{W_t}{W_i} = \frac{4\beta}{(1 + \beta)^2} \doteq 1.1 \times 10^{-3} \; . \qquad (2.114)$$

The pressure level underneath the water surface is thus about 6 dB higher than its value in the air above the surface. However, the corresponding intensity and power levels are about 29.5 dB lower. It follows that, while airborne sources transmit very little acoustic power into water, they may be as detectable by pressure-sensitive transducers as by microphones.

Young (1973) has demonstrated that the entire sound field in water attributable to a source in air can be calculated with reasonable accuracy by locating a virtual source of strength 7 dB less than the actual source directly under the actual source at one-fifth its elevation, and assuming a dipole (cosine) radiation pattern with its maximum directly under the source.

Transmission from Water into Air

Detection of sources in water by instruments in air is a different matter. Again considering near-normal incidence,

$$\beta \doteq \frac{3600}{\sin \theta_i} \geqslant 3600 \; , \qquad (2.115)$$

and the power transmission ratio is

$$\frac{W_t}{W_i} = \frac{4\beta}{(1 + \beta)^2} \doteq \frac{4}{\beta} \leqslant 1.1 \times 10^{-3} \; . \qquad (2.116)$$

Thus the loss is at least 29.5 dB, the same as for transmission from air to water. However, the relative pressure level is much lower since, from Eq. 2.108, the pressure ratio is

$$\alpha_t = \frac{2}{1 + \beta} \leqslant \frac{2}{3601} = 5.5 \times 10^{-4} \; , \qquad (2.117)$$

and it follows that the pressure transmission loss is at least 65 dB. Most noise sources in water are therefore virtually undetectable in air; only large high-power sources such as active sonars and explosions produce significant signals in the atmosphere.

To calculate the pressure field in air from a source in water, Young (1973) placed a virtual source at five times the actual source depth, of strength 52 dB less than the actual source level, and assumed a cosine-squared directional pattern. This calculation confirms the low levels in air indicated by Eq. 2.117.

Reflection of Underwater Sound by Ocean Surfaces

The air-water interface at the ocean surface is an excellent reflector of underwater sound. From Eq. 2.105,

$$\alpha_r = -\frac{\beta - 1}{\beta + 1} \doteq -\left(1 - \frac{2}{\beta}\right) \doteq -0.9995\ , \tag{2.118}$$

showing that reflection occurs virtually without loss of amplitude and with a phase shift of 180°. Since the angle of reflection equals the angle of incidence, the effect of a free surface can be treated by considering negative image sources above the surface, as illustrated in Fig. 2.3. Often the geometrical situation is such that a receiver receives sound by both direct and reflected paths, resulting in complicated interference patterns. This subject is considered in more detail in Section 4.6.

The result given by Eq. 2.118 has been derived for a smooth plane surface. The actual ocean surface is of course quite rough, causing scattering of incident sound. Scattering may be thought to cause the image to dance around, or it may be considered to cause reflection loss and thus to reduce the amplitude of a fixed image. In any case, the effect increases with frequency, wave height and angle of incidence. A surface can be considered to be acoustically smooth and scattering loss to be negligible provided

$$\frac{h \sin \theta_i}{\lambda} < \frac{1}{4}\ , \tag{2.119}$$

where h is wave height. Thus, at 1 kHz, for which $\lambda = 1.5$m, at an angle of 10° the ocean is acoustically smooth for wave heights up to at least 2 m.

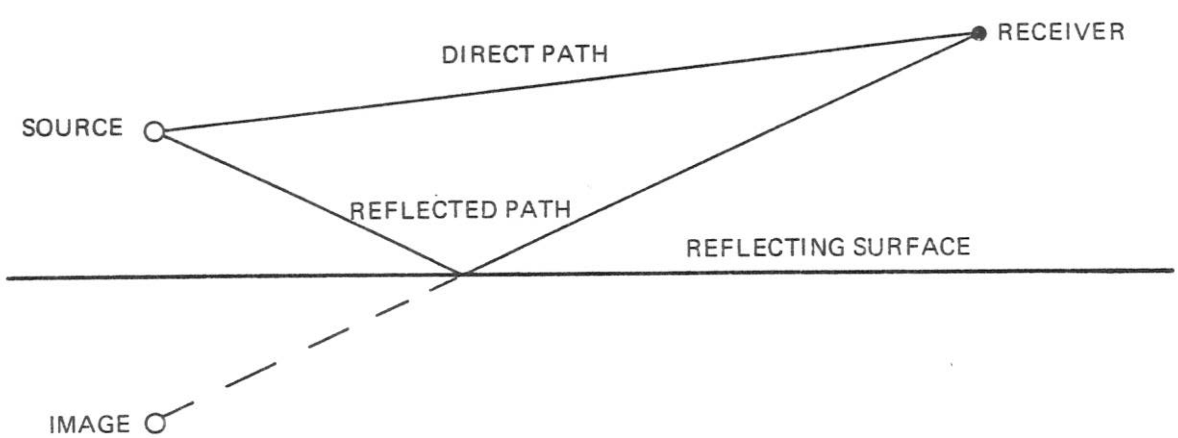

Fig. 2.3. Image Source for Reflection by Air-Water Interface

2.6 Finite-Amplitude Effects

In all of the discussions of sound waves in the present chapter, linearity has been assumed. Resolution of waves into independent harmonic components depends on this property. The derivation of the wave equation assumes linearity. Most phenomena in underwater acoustics can be completely understood in terms of linear acoustics. However, there are occasions when non-linear, finite-amplitude effects must be considered. The present brief exposition is intended merely to call attention to this aspect of acoustics and to present criteria which can be used to assess the possible significance of non-linear effects.

In linear acoustics, a fundamental assumption is that the particle velocity is small compared to the speed of sound. It follows that all parts of a wave travel at the same speed. Actually, this is not strictly true. The actual speed is a superposition of the local sound speed and the local particle velocity, both of which may vary with place in the wave. Thus, locally,

$$\frac{dx}{dt} = c + v' = c_o + \rho_o c_o \left(\frac{dc}{dp}\right) v' + v' . \tag{2.120}$$

The occurrence of non-linear effects is thus primarily attributable to the finite amplitude of the particle speed relative to that of sound, and secondarily to the fact that the speed of sound is itself not a constant. Beyer (1960) showed that the degree of non-linearity is controlled by the parameter

$$\frac{1}{c_o}\left|\frac{dx}{dt}\right|_{max} - 1 = \left(1 + \rho_o c_o \frac{dc}{dp}\right) M' = LM' , \tag{2.121}$$

where M' is the acoustic Mach number of the wave at its peak. For a gas, L can be shown to be related to the ratio of the specific heats, and to be equal to about 1.2 in air. For water, the value is about 3.5. Based on the larger value of L, one might expect greater non-linear effects in water than in air. However, the acoustic Mach number is invariably much less in water than in air.

Non-linear effects are also strongly frequency dependent, since the effects cumulate as the wave travels a number of wavelengths. A parameter governing non-linearity is the stretched range variable, σ, defined by

$$\sigma \equiv LM' k r_o \ln \frac{r}{r_o} , \tag{2.122}$$

where r_o is a reference distance, usually about one meter. Expressing the acoustic Mach number in terms of the peak acoustic pressure,

$$\sigma = L\omega \frac{P(r_o) \cdot r_o}{\rho_o c_o^3} \ln \frac{r}{r_o} . \tag{2.123}$$

Distortion of the wave is negligible as long as σ is less than about 0.15. It follows that non-linear effects can be completely ignored in sea water provided that the product of source pressure level and frequency do not exceed 30 kilohertz-atmospheres.

The only sources which approach non-linearity in normal sea water are large active sonars and explosions. However, if the medium contains quantities of bubbles, as in a zone of cavitation, L may be significantly higher than normal and c_o significantly lower, resulting in non-linear effects at much lower sound pressures. Thus, harmonic distortion, intermodulation products and increased absorption can all be expected to occur when the compressibility of a liquid is significantly increased due to the presence of bubbles.

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