CHAPTER 1

INTRODUCTION

1.1 Noise: Unwanted Sound

Usually when a person uses the word *noise* he is referring to sounds such as those of jet aircraft flying overhead, the rumble of trucks from a nearby highway, or the racket being made by his neighbor's children. These sounds are annoying because they intrude on him and interfere with activities such as conversation and sleep; they may even interfere with his ability to think. *Noise is unwanted sound that interferes with the normal functioning of a system.* The seriousness of the noise and the degree to which noise-control measures are required depend not so much on the level of the noise as on the amount of interference it causes with other functions.

Underwater Noise

Underwater noise is sound in water that limits the military effectiveness of naval systems. Submarines are particularly prone to experiencing such limitations, since sounds which they radiate can reveal their presence to an enemy. In addition, they depend upon acoustic signals for communications and use sonar to detect the presence of any enemy, which functions are also limited by noise.

Submarines are not the only naval systems for which noise plays a vital role in limiting ability to perform assigned functions. Sounds radiated by surface ships reveal their presence to enemy submarines; and, like the submarine, sonar self-noise limits their ability to detect targets. In some cases, sounds radiated by one surface ship may even interfere with sonar performance on another. Another example of limitation by self-noise is that of passive acoustic homing torpedoes which use sounds radiated by ships and submarines to locate these targets. Finally, the effectiveness of otherwise quiet systems, such as buoys, may be determined by the ambient noise background of the sea.

That underwater noise plays a dominant role in naval warfare is today recognized in most Navy circles. Appreciable efforts are devoted both to reducing noise and to developing methods to exploit it. Of necessity, much of this work is classified. However, the phenomena involved are related to topics in physics and mechanics and can be discussed in a general way without divulging classified aspects of specific military systems.

Noise Is Unavoidable

According to the Second Law of Thermodynamics, no useful mechanical process can take place without generating some heat. If heat were not produced, it would be possible to create a perpetual motion machine. It is not as well recognized, but it is probably equally true that no useful mechanical process can occur without generating some vibration and therefore at least a little noise. Thus, noise is also an unavoidable by-product of machines. Associated with each steady, work-producing force there are always small unsteadinesses, or vibrations, and these vibrations are transmitted to the surfaces of the machine, from which they radiate as sound. Likewise, when a body moves through a fluid, turbulent motions are created. Not only do these turbulent motions eventually decay into heat, but also they radiate a small amount of sound. Only in a vacuum would it be possible to do useful work without producing sound. The acoustician may well be tempted to modify the well-known Second Law of Thermodynamics to include sound along with heat as necessary by-products of mechanical processes.

The amount of sound power radiated into air by various mechanisms varies from as low as a microwatt for a very small fan to many kilowatts for airplanes and over a megawatt for a large rocket. Power levels in water tend to be much lower. A modern submarine proceeding at slow speed produces on the order of 10 mW acoustic power, while surface ships generally radiate from five to 100 Watts. When one realizes that mechanical powers of the order of many thousands of horsepower are involved in operating ships and submarines, it is apparent that only a very small fraction of this mechanical power is actually converted into underwater sound.

Although power levels radiated into water by ships, submarines and torpedoes are relatively low, this does not mean that radiated underwater noise is of no consequence. Sources that radiate as much as one Watt of acoustic power can be detected at relatively long ranges by modern passive sonars. The same power in air might carry only several blocks. The reason for this difference is that in water relatively high acoustic pressures are associated with low power levels; since detection systems respond to acoustic pressures rather than to power density (intensity), it is pressure levels rather than power levels that determine the detectability of underwater sounds.

Acoustic Conversion Efficiency

A useful concept in analyzing noise mechanisms is that of acoustic conversion efficiency, defined as the ratio of the sound power radiated to the mechanical power of the source:

$$\eta_{ac} \equiv \frac{\text{Acoustic Power}}{\text{Mechanical Power}} = \frac{W_{ac}}{W_{mech}} . \tag{1.1}$$

This ratio finds its greatest use in sorting out noise sources, since different sound-producing mechanisms have different relationships for their acoustic conversion efficiencies.

Acoustic conversion efficiencies are much lower in water than they are in air. Conversion efficiencies as low as 10^{-8} are common for sources in water, while values as high as 10^{-4} to 10^{-2} are often found for sounds radiated into air. This difference between the two media is caused by their relative compressibilities: water is much less compressible than air. Since it is the compressibility of a medium that makes sound possible, the same mechanical power generates more sound power in air than it does in water.

The parameter that measures the relative importance of compressibility is the Mach number, M, defined as the ratio of a pertinent mechanical speed to the speed of sound waves:

$$M \equiv \frac{U}{c} . \tag{1.2}$$

If a medium were totally incompressible, its speed of sound would be infinite, and the Mach number would always be zero. Thus, although water is for many purposes practically incompressible, it does have slight compressibility; and Mach numbers, though low, are finite. In

Chapter 3, which covers basic sound radiation mechanisms, it will be shown that acoustic conversion efficiencies can usually be expressed as functions of Mach number of the form

$$\eta_{ac} \sim M^n \tag{1.3}$$

where the exponent n is equal to or greater than unity. The low acoustic conversion efficiencies found in underwater sound are related to relatively low values of the Mach number in water.

In dealing with many noise sources, it is useful to divide the noise-production process into three parts: generation of a vibratory motion, transmission of this vibration to a radiating surface, and radiation of sound into the medium. The acoustic conversion efficiency can thus be expressed as the product of three conversion efficiencies, one for each of the three processes:

$$\eta_{ac} = \eta_{vibr} \cdot \eta_{trans} \cdot \eta_{rad} \; . \tag{1.4}$$

It is the last term, the radiation efficiency, which is controlled by the Mach number and which differs most between air and water. The other terms are usually independent of the fluid medium. Obviously, the over-all acoustic conversion efficiency is always less than the radiation efficiency.

Noise Control

It is not practical, economical or even desirable to attempt to eliminate all noise from mechanical systems. Noise control is the technology that evaluates the need for noise reduction and then attempts to achieve acceptable noise levels in a manner consistent with economic and operational considerations. An understanding of basic noise mechanisms is essential to successful noise control.

Equation 1.4 serves as a useful guide to the principles used in noise control. Noise reduction can be accomplished in three different ways:

1. by reducing the fraction of the source mechanical power converted into vibratory power, or by selecting machinery with lower rated mechanical powers;
2. by isolating the source from radiating surfaces; i.e., by reducing the efficiency of vibration transmission; or
3. by reducing the radiation efficiency of the radiating surfaces.

Of the three, the second, isolation of the source from the radiating surface, is generally the most easily accomplished. Reduction of noise at a source often requires redesign of a mechanical system, and reducing the radiation efficiency may require extensive application of anti-radiation coatings. Although noise reduction *per se* is not the purpose of the present volume, many topics pertinent to noise control are considered; the reader may expect to gain some understanding of methods of noise reduction.

Types of Underwater Noise

There are a number of different manifestations of underwater noise. While consistent definitions for all of the principal types encountered in naval systems have not been universally adopted, the following definitions are consistent with those adopted by the American Standards Association:

Radiated Noise – noise radiated into the water that can be used by a passive listening sonar to detect the presence of a vehicle at a considerable distance.

Ambient Noise – all noises associated with the medium in which a sonar operates that would exist in the medium if the sonar platform or vehicle itself were not present.

Platform Noise – that noise measured by a single, omnidirectional, platform-mounted hydrophone in the presence of an operational platform. Conceptually, platform noise should be simply noise attributable to the presence of the platform, but actual measurements of platform noise invariably include the contribution of ambient ocean noise.

Sonar Self-Noise – noise associated with a platform and its sonar hydrophones and preamplifiers, as measured through the sonar hydrophone array.

Sonar Background Noise – all noise at the output of a sonar array that limits the detection of signals by a signal processor. Sonar background noise includes the contribution of the medium as well as platform noise and any noises contributed by hydrophones, cables or preamplifiers. (Actually, most sonar self-noise measurements are really background noise measurements, since such measurements are generally made under circumstances that do not permit separate measurement of ambient noise and there is no practical method to estimate the contribution of the medium.)

1.2 Decibels and Levels

Acoustic measurements are almost invariably expressed in decibels, which are units involving logarithms of various ratios. The need for a logarithmic measurement unit arose in acoustics for two reasons: first, the range of sound intensities found in practice varies from about 10^{-9} W/m^2 for a barely intelligible whisper to over a kW/m^2 near a jet aircraft; second, human response to acoustic stimuli is approximately logarithmic. For these reasons, it seemed logical to adopt logarithmic units for acoustic measurements. Use of logarithmic measures is not unique to acoustics; such quantities are quite common in thermodynamics. Problems have arisen in acoustics due to inconsistent choices of reference quantities.

Decibels

Decibels were originally defined in the early 1920's by workers in the electrical communications industry who were interested in the power transmission capability of networks. They noted that as long as a network was linear the output power maintained a constant ratio to the input power. Since these ratios were often quite large, they expressed them by a logarithmic quantity:

$$\textit{transmission ratio} = \log_{10} \frac{W_2}{W_1} . \tag{1.5}$$

The unit was named *Bel* after Alexander Graham Bell, inventor of the telephone. To avoid dealing in fractions of Bels, a unit one-tenth as big was chosen, namely, the *decibel*:

$$\textit{trans. ratio in dB} = 10 \log \frac{W_2}{W_1} . \tag{1.6}$$

When high impedance networks became common, interest switched to voltage ratios rather than power ratios. Since, for constant resistance, power is proportional to the square of the voltage, twenty times the logarithm to the base ten was chosen for voltage ratios,

$$\textit{trans. ratio in dB} = 20 \log \frac{e_2}{e_1} . \tag{1.7}$$

thus maintaining constancy of numerical values of transmission ratios when expressed in decibels.

Transmission Loss

In acoustics, intensity is a power-like quantity and pressure corresponds to voltage. When the use of decibels was extended from electric networks to acoustics, it was logical to define acoustic transmission ratios by

$$\text{trans. ratio in dB} = 10 \log \frac{I_2}{I_1} = 20 \log \frac{p_2}{p_1} . \tag{1.8}$$

Actually, in underwater sound it is more common to express transmission ratios as transmission losses, since pressures and intensities usually decrease with increasing distance from a source. Assuming position one to be closer to the source, the *transmission loss in dB* is defined by

$$TL = 20 \log \frac{p_1}{p_2} , \tag{1.9}$$

where the pressures are usually root-mean-square (rms) values. While invariably positive, transmission loss is usually plotted in a negative sense since received signals decrease as transmission loss increases.

Levels

Use of decibels in acoustics causes no concern when the application involves comparison of intensity or pressure close to a source with that at a distant measurement point, as in transmission loss. Some confusion has arisen, however, from the practice of expressing quantities measured at a single location in terms of their decibel values, called *levels*. The intensity and pressure at a point are expressed as levels by taking logarithms of their ratios to reference values,

$$IL \equiv 10 \log \frac{I}{I_o} , \tag{1.10}$$

$$SPL \equiv 20 \log \frac{p}{p_o} , \tag{1.11}$$

where I_o and p_o are reference values. This procedure is in itself straightforward. However, problems have arisen in the selection of reference quantities, especially for underwater acoustics.

Reference Pressures

It would seem logical to write a pressure level simply as $20 \log_{10} p$, where p would be measured in Newton/m^2 if one were using the MKS system, or in dyne/cm^2 if one were using cgs units. The problem is that most measured acoustic pressures are smaller than 1 N/m^2 or 1 dyne/cm^2, and the corresponding levels would be negative. The early producers of sound level meters wanted their decibel readings to be positive. They, therefore, sought to measure intensity and pressure not relative to unity in cgs units but relative to a value small enough to assure positive levels. In the early 1930's, a number of references were proposed, varying from 2×10^{-4} to 1.4×10^{-2} dyne/cm^2.

In 1932, a subcommittee of the newly formed American Standards Association Sectional Committee Z24 for Acoustics tackled the problem of finding a standard reference for noise measurement. Its deliberations were strongly influenced by a desire to consider intensity as a fundamental quantity. It chose 10^{-16} W/cm^2, which equals 10^{-12} W/m^2, as the primary reference. However, intensity is seldom measured directly; it is usually inferred from a pressure measurement. For unidirectional plane and spherical waves, acoustic intensity and pressure are related by

$$I = \frac{\overline{p^2}}{\rho_o c_o}, \tag{1.12}$$

where ρ_o is the density and c_o the speed of sound of the medium. For standard air, pressure levels will equal intensity levels if the reference pressure is taken to be 0.000204 dyne/cm^2. Since the difference is small, this has been rounded off to 0.0002 dyne/cm^2, which is now the reference for all pressure measurements in airborne acoustics.

While the selection of 0.0002 dyne/cm^2 had validity for airborne sound, it had no physical significance for other media. While many workers in underwater acoustics adopted this reference, others chose 1 dyne/cm^2. This use of two references by different groups has lasted until very recently. Only in the past several years has the underwater sound community agreed on a single pressure reference. It is interesting to trace the history of reference pressures in underwater acoustics and to see how a new third unit came to supplant two established ones.

When underwater noises of ships and other vehicles were first measured, workers used existing noise measuring gear already calibrated relative to 0.0002 dyne/cm^2. Graphs were simply marked "sound pressure level in decibels." However, groups involved in transducer calibration during World War II desired a larger unit. Noting that the then standard pressure had no physical significance relative to intensity in water, they chose 1 dyne/cm^2, often called one microbar, as their reference pressure. Use of this reference spread, and by the end of WWII about half of the community was using each standard.

An attempt at standardization after WWII failed, and the situation continued for about 20 years. It might have continued indefinitely, but naval personnel were making increased use of acoustic data, and presentation of such data using two different references caused much unnecessary confusion. In 1961, Writing Group S1-W44 was appointed by the American Standards Association at the request of the U.S. Navy Bureau of Ships to recommend a reference sound pressure for underwater acoustics. A survey of the community revealed a near 50-50 split, and neither side was willing to concede. After several years of debate, support gradually developed for the idea of adopting a new fundamental unit to be used in the MKS system and small enough that all measured levels would be positive. The reference pressure ultimately recommended was 10^{-6} N/m^2, called a *micropascal* and abbreviated μPa. In 1970, by order of the Chief of Naval Operations, this unit was adopted by the U.S. Navy, and it is rapidly becoming the standard for all underwater measurements. All numerical values presented in the present volume are referenced to this unit.

Since the new standard reference pressure is so new, many currently used texts and reports use the old references. Values relative to 1 dyne/cm^2 can be converted to micropascals by simply adding 100 dB, while values referenced to 0.0002 dyne/cm^2 require the addition of 26 dB.

Source Level

Acoustic noise measurements are sometimes presented as measured, but more often they are presented in terms of constructs that are derived from actual measurements through certain assumptions. One of these constructs is *source level.* It would be convenient if the total acoustic power radiated by a source were itself measurable, but power is not a directly measurable quantity. What is usually measured is acoustic pressure at some distance from the source. Since pressure varies with distance, pressure alone is not a unique measure of source strength. Some attempts have been made to standardize measurement distances: 3 and 10 feet from machines in air, and 20 and 100 yards or meters from ships and submarines. However, it is not always possible to make measurements at standard distances. Source level is a construct which enables measurements made at a variety of distances to be used as comparable indicators of source strength.

Source level is defined as the pressure level that would be measured at a reference distance of one foot, one yard or one meter from an ideal point source radiating the same amount of sound as the actual source being measured. Since most practical sources have directional radiation patterns, source level is properly a function of direction. Source levels are never measured directly. Rather, they are inferred from measurements at greater distances. The complete specification of source level includes the reference distance, thus:

dB re 0.0002 dyne/cm² at 1 foot (in air), and

dB re 1 μPa at 1 meter or 1 yard (in water).

The concept of source level as a measure of the strength of a noise source has become entrenched in underwater acoustics largely because of its role in sonar performance prediction.

Power Level

While not usually directly measurable, acoustic power is often calculated directly in theoretical equations. It can be expressed as a logarithmic quantity, called *power level,* by

$$PWL \equiv 10 \log \frac{W_{ac}}{W_o} , \qquad (1.13)$$

where the reference power, W_o, is usually 1 pW (10^{-12} W) in airborne acoustics and 1 W in underwater sound. Power levels and source levels are related in a simple manner only when sources radiate uniformly in all directions. In air the relationship is

$$PWL \doteq L_S , \qquad (1.14)$$

where L_S is the source level in dB re 0.0002 dyne/cm² at 1 ft and *PWL* is power level in dB re 10^{-12} W. In water, the relationship is

$$PWL \doteq L_S - 171 , \qquad (1.15)$$

where L_S is in dB re 1 μPa at 1 yd or 1 m, and power level is relative to 1 W.

Spectrum Level

When dealing with single-frequency tonal components, source levels refer to the total pressure of the signal. The situation is not so simple when dealing with broadband sources, since the measured level is then a function of filter width. Since different measurement activities use different filter widths, their results are not directly comparable. In order to be able to compare and average values obtained using different filters, the concept of *equivalent spectrum level* is used. This is defined as the level that would have been measured using an ideal 1-Hz filter. Measured spectra are readily converted to spectrum level if the distribution of energy is relatively uniform throughout the measurement band. In this case

$$L_s = L_S - 10 \log \Delta f \,, \tag{1.16}$$

where Δf is the filter bandwidth and a lower case subscript refers to spectrum level, while upper case continues to represent total or band level. Even when a spectrum is not flat, Eq. 1.16 is used as the definition of equivalent spectrum level. Thus, calculated equivalent spectrum levels are measures of average levels within a filter band.

It is common practice to plot each spectrum level at the *effective center frequency* of the filter, defined as the geometric mean of the upper and lower cut-off frequencies:

$$f_c = \sqrt{f_u f_\ell} \,. \tag{1.17}$$

Calculated spectrum level is a good measure of the actual spectrum level at the filter center frequency provided the spectrum is continuous and does not change much within the filter band. It can be shown by integrating over a filter band that this procedure is valid provided actual levels do not vary by more than about 9 dB within the band. If the variation is greater than this, errors in excess of a decibel will be introduced.

One-third-octave filters are widely used today. While it is common practice to convert one-third-octave data to spectrum level, some measurement groups prefer presenting their results as actually measured, rather than making the conversion. The reason is that many actual spectra contain tonal components, and spectrum levels can then be quite misleading. Spectrum levels are distinguished from band levels by adding an *s* to *dB*. Thus, *dBs* means "dB in a 1-Hz band," not the plural of dB.

Decibel Arithmetic

When using decibels, equations that otherwise would have involved multiplication or division of numbers become simple additions and subtractions. Difficulties occur, however, when performing operations that would be additions or subtractions of linear functions. Thus, the addition or subtraction of sounds from several sources requires transformation of the decibel values to linear values by use of antilogs and transformation back to logarithmic values after performance of the addition or subtraction. Results of source addition and subtraction also depend on whether the several signals are at the same single frequency, or are either broadband or at different frequencies. In the first case, the signals are coherent; otherwise they are incoherent. Formulas for coherent and incoherent decibel arithmetic are given in Appendix B.

1.3 Significance of Spectra

In recent years, emphasis in noise measurements has increasingly been placed on narrowband spectra. There are several reasons for this trend. Most important is the fact that its detailed spectral distribution is a most important clue as to the nature of a noise source. Before noise reduction measures can be applied, it is essential that dominant sources be identified. Narrowband spectral analysis is a potent tool in diagnostic noise studies.

The reason that spectral analysis enables classification of noise sources is that, through the Fourier transform, there is a direct relationship between the shape of a signal in the time domain and its spectrum. The simplest signal to contemplate is a pure sine wave, of frequency f_o. In frequency space, the spectrum of a pure sine wave is a spike of near-zero width at frequency f_o. Nearly pure sine waves are generated by mechanical unbalance forces of rotating machinery running at constant speed. In practice, slight variations in speed usually modulate the frequency, thereby producing tones with finite width in spectral space. The spectral width of a tone is a direct measure of source stability. It can be expressed in terms of the Q of the tone, defined as the ratio of its frequency to its bandwidth,

$$Q \equiv \frac{f_o}{\Delta f} \,, \qquad (1.18)$$

where Δf is the half-power bandwidth.

Some sources produce noise through repeated impacts, with each impact generating a sharp pulse in the time domain. The resultant spectrum consists of a large number of tonal components separated in frequency by the fundamental pulse repetition frequency, the number of strong tonal components being a function of the decay time of the individual pulses. Since time between impacts is never exactly constant, individual tones have finite spectral widths which are dependent on source stability. Other noise sources, notably cavitation, produce noise through impulses which are random both in time of occurrence and in amplitude and duration. They produce continuous spectra having some energy at all frequencies, and generally having a broad spectral peak at a frequency which is related to the most prevalent decay time.

The important point is that characteristics of a spectrum are directly related to the nature of the phenomena producing them, and that through detailed spectral analysis one can deduce a great deal about noise sources. In the present volume, as each type of source is discussed, the nature of its spectrum will be developed and any special characteristics noted.

1.4 Passive Sonar Equation

The *sonar equation* is an expression in decibels of signal-to-noise relationships for a passive sonar. Figure 1.1 is a simplified schematic of a passive sonar system, showing its essential elements. The *signal level* in the water at the hydrophones of a receiving sonar can be expressed by

$$SL = L_S - TL \,, \qquad (1.19)$$

provided the reference distance for transmission loss is taken to be the same as that used in defining the source level. Thus, in underwater sound, *transmission loss* at distance r is defined by

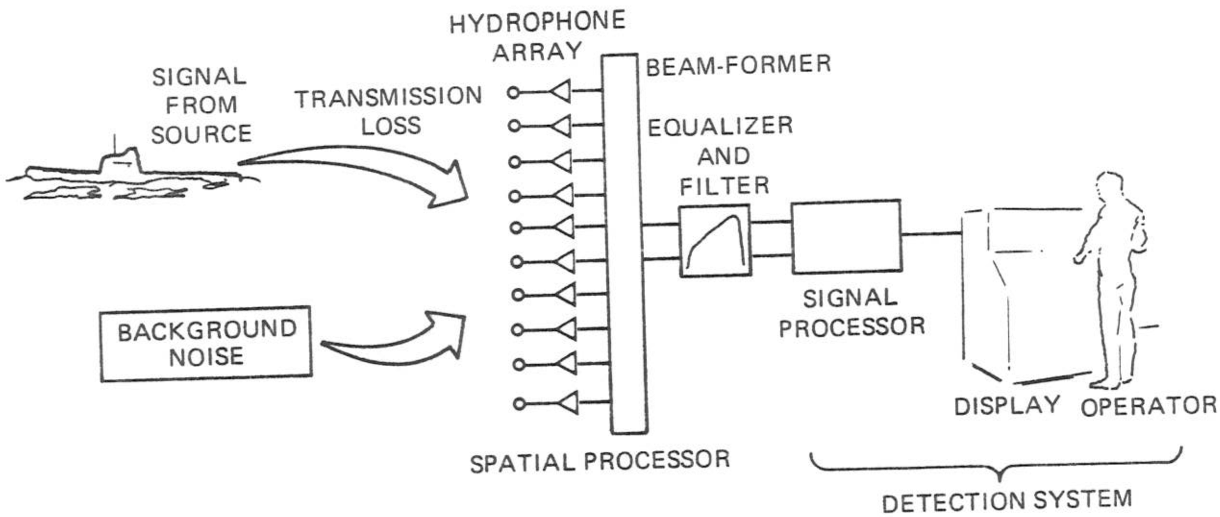

Fig. 1.1. Passive Sonar System

$$TL \equiv 20 \log \frac{p(1)}{p(r)} , \tag{1.20}$$

where $p(1)$ is the rms pressure that would be measured one yard or one meter from an ideal point source. Representing the *background noise level* in the water by L_N, the *signal-to-noise ratio* at a hydrophone is

$$\left(\frac{S}{N}\right)_{water} \equiv SL - L_N = L_S - TL - L_N . \tag{1.21}$$

The array of a sonar is a spatial processor that discriminates against background noise. It improves the signal-to-noise ratio by an amount equal to the *array gain*, AG. The output of an array is the input to the signal processor, so that

$$\left(\frac{S}{N}\right)_{in} \equiv \left(\frac{S}{N}\right)_{water} + AG = L_S - TL - \left(L_N - AG\right) . \tag{1.22}$$

The level of signal-to-noise into the signal processor for which the probability of detection is 50% is called the *detection threshold*, DT, or *recognition differential*, N_{RD}, of the processor. Detection is more likely whenever the signal-to-noise ratio into the processor exceeds the detection threshold, and is less likely when it is lower. The difference is called *signal excess*:

$$SE \equiv \left(\frac{S}{N}\right)_{in} - N_{RD} = L_S - TL - \left(L_N - AG\right) - N_{RD} . \tag{1.23}$$

This equation for signal excess is one form of the passive sonar equation.

The passive sonar equation is relatively simple, but its application is complex. One problem is that most sonars are broadband sonars, and quantities such as array gain and transmission loss are not constant across a frequency band. Another problem arises from effects of temporal and spatial fluctuations of various quantities. Still another difficulty is that the effective transmission loss for a directional receiver may differ from the omnidirectional value that is usually measured. Normally, the sonar equation applies to a relatively short sample time, yet often what is wanted is a prediction of detection performance over long periods. Despite these difficulties in its application, the sonar equation is nevertheless the basic framework used in treating topics in underwater acoustics.

1.5 Some Mathematics

While emphasis in the present volume is on physical principles and on understanding fundamental mechanisms whereby noise is generated, a number of mathematical derivations are included and equations are used extensively to describe functional relationships. The following paragraphs discuss a number of mathematical concepts and also serve to indicate the nomenclature used.

Scalars, Vectors and Tensors

Physical quantities are classified as scalars, vectors or tensors, according to their dependence on direction.

Scalars are physical quantities that are fully described by numbers and are independent of direction. Examples are temperature, density, speed and energy.

Vectors are physical quantities that involve direction as well as magnitude. Examples are velocity, force and momentum. Vectors are indicated by arrows over symbols. In cartesian coordinates

$$\vec{A} = \hat{\imath} A_x + \hat{\jmath} A_y + \hat{k} A_z \,, \tag{1.24}$$

where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors in the x, y and z directions. The magnitude of a vector is related to its components by

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \,. \tag{1.25}$$

A *tensor* of the second rank is a physical quantity whose full description requires specification of two directions. Examples are mechanical strain and stress, for which both the direction of a surface and that of a force must be specified. Such quantities can be written in cartesian coordinates in terms of nine independent components, each of which is given a double subscript to account for the two directions.

In the most general sense, all physical quantities are described by tensors of various ranks. A vector is a tensor of first rank, while a scalar is a tensor of zero rank. Second rank tensors are sometimes called *dyadics*.

Tensor Notation

In the present volume, a mixture of vector and tensor notation is used in such a way as to attempt to convey the physics being described by the equations. In tensor notation, directions are represented by subscript indices and quantities are summed for all values of the index whenever the same index appears twice in a single term of an equation. The rank of the tensor is indicated by the number of indices required to specify it. Thus, the components of a vector are represented by a symbol having a single subscript, as A_i. Tensor notation is most readily interpreted when using cartesian coordinates, for which the subscript indices represent the x, y and z directions in sequence.

A useful symbol in tensor notation is the *Kronecker Delta* symbol. This is a delta with two subscript indices. When the two indices are the same, its value is unity; otherwise it is zero. Using this symbol,

$$\delta_{ij} \frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} . \tag{1.26}$$

When several tensor quantities of various rank are combined, the rank of the resulting combination is equal to the number of indices that do not appear twice. Thus Eq. 1.26 is a scalar equation, since each index appears twice.

Vector Operations

Vectors may be added, subtracted, multiplied and differentiated, but the rules differ somewhat from the corresponding operations for scalars.

Vector addition and subtraction are performed by carrying out the specified operation on each of the components:

$$\vec{A} \pm \vec{B} = \hat{i}\left(A_x \pm B_x\right) + \hat{j}\left(A_y \pm B_y\right) + \hat{k}\left(A_z \pm B_z\right) . \tag{1.27}$$

Scalar multiplication of two vectors produces a scalar having its value equal to the product of their magnitudes times the cosine of the angle between them:

$$\vec{A} \cdot \vec{B} = A_i B_i = A_x B_x + A_y B_y + A_z B_z = AB \cos(A,B) . \tag{1.28}$$

The scalar product is sometimes called the *dot product.*

Vector multiplication of two vectors yields a vector perpendicular to their plane in the direction of a right-handed screw, having its magnitude equal to the product of their magnitudes times the sine of the angle between them:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\left(A_y B_z - A_z B_y\right) + \ldots . \tag{1.29}$$

$$|\vec{A} \times \vec{B}| = AB \sin(A,B) . \tag{1.30}$$

Since direction depends upon order,

$$\vec{B} \times \vec{A} = -\left[\vec{A} \times \vec{B}\right]. \tag{1.31}$$

The vector product is often referred to as the *cross product.*

The *derivative* of a vector is a vector having as its components the derivatives of the individual components:

$$\frac{d\vec{A}}{ds} = \hat{i}\frac{dA_x}{ds} + \hat{j}\frac{dA_y}{ds} + \hat{k}\frac{dA_z}{ds} . \tag{1.32}$$

Derivatives of scalar and vector products are obtained by applying the product rule for differentiation:

$$\frac{d(\vec{A} \cdot \vec{B})}{ds} = \vec{A} \cdot \frac{d\vec{B}}{ds} + \vec{B} \cdot \frac{d\vec{A}}{ds}, \qquad (1.33)$$

$$\frac{d[\vec{A} \times \vec{B}]}{ds} = \left[\vec{A} \times \frac{d\vec{B}}{ds}\right] + \left[\frac{d\vec{A}}{ds} \times \vec{B}\right]. \qquad (1.34)$$

Vector Operators

There are three vector differential operators that are used extensively in physics when dealing with field quantities defined over a region of space:

The *gradient* of a scalar is a vector having the magnitude and direction of the greatest space rate of change of the scalar:

$$grad\ \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}. \qquad (1.35)$$

The components of the gradient are the rates of change in each direction. The symbol ∇ is commonly used to represent the gradient vector differential operator:

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}. \qquad (1.36)$$

It can be applied to vectors as well as to scalars.

The *divergence* of a vector is a scalar obtained by taking the scalar product of the gradient operator and the vector:

$$div\ \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \qquad (1.37)$$

It represents the net outward flow of a quantity from a differential volume.

The *curl* of a field vector is a vector giving the magnitude and direction of its *rotation*. It is obtained by taking the cross product of the gradient operator and the vector:

$$curl\ \vec{A} \equiv \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \ldots. \qquad (1.38)$$

Since the curl of a vector is a measure of its rotation, vector fields having zero curl are termed *irrotational fields*.

Scalar Potentials

Many fluid flows, in acoustics as well as in fluid mechanics, are irrotational. Whenever the curl of a vector quantity is zero, it is possible to define that vector quantity in terms of the gradient of a *scalar potential*,

$$\vec{A} = \pm\ grad\ \phi\ , \tag{1.39}$$

where the sign is arbitrary, but is generally taken as negative. It can readily be shown that existence of a scalar potential implies irrotationality, since the curl of a gradient operator is always zero. Potentials are used frequently, since they allow irrotational vector fields to be treated in terms of scalar fields, thereby essentially replacing the three equations for the three vector components by a single equation.

In many instances the differential equation defining a scalar potential is of second order, involving the divergence of the gradient of the potential. This second-order differential operator is called the *Laplacian* and is represented by ∇^2:

$$\nabla^2 \phi = div\ grad\ \phi = \nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2\phi}{\partial x_i^2}\ . \tag{1.40}$$

The Laplacian operator plays a central role in equations of acoustics.

Spherical Symmetry

Thus far the various vector operators have been described in cartesian coordinates. However, many problems of fluid mechanics and acoustics exhibit spherical symmetry and are better treated in spherical coordinates. Spherical coordinates involve a radial unit vector, $\hat{r}$, and two angular coordinates. When spherical symmetry exists, spatial derivatives with respect to all directions except the radial direction are zero, and the vector operators take on relatively simple forms:

$$\vec{A} = \hat{r} A_r(r)\ , \tag{1.41}$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\ . \tag{1.42}$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\,\frac{\partial(r^2 A_r)}{\partial r} = \frac{\partial A_r}{\partial r} + \frac{2}{r} A_r\ . \tag{1.43}$$

$$\nabla^2\phi = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right) = \frac{1}{r}\,\frac{\partial^2(r\phi)}{\partial r^2}\ . \tag{1.44}$$

Line, Surface and Volume Integrals

Line integrals of a function are carried out between two points along a specified path. They are written

$$\int_A^B f ds\ ,$$

where ds is a differential segment of the specific path. If the quantity is a vector, then the line integral is the integral of the component of the vector in the direction of the segment, given by

$$\int_A^B \vec{f} \cdot \vec{ds} = \int_A^B \left(f_x dx + f_y dy + f_z dz\right) = \int_A^B f_i ds_i \ . \tag{1.45}$$

In general, line integrals depend upon the specific path chosen. However, for the important class of fields for which rotation is zero and for which a scalar potential exists, the integral is independent of path and dependent only on its end points:

$$\int_A^B \nabla\phi \cdot \vec{ds} = \int_A^B \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz\right) = \int_A^B d\phi = \phi_B - \phi_A \ . \tag{1.46}$$

A *contour integral* is a line integral taken around the edges of a surface, returning to the starting point. The value of a contour integral is a measure of the rotation enclosed by the contour and is zero for irrotational fields.

Differential elements of surface are described by a magnitude and by the direction of the outward-drawn normal to the surface. The *surface integral* of a vector function is the integral of its normal component over the surface:

$$\int_S \vec{f} \cdot \vec{dS} = \int_S \left(\vec{f} \cdot \hat{n}\right) dS = \int_S f_n dS \ . \tag{1.47}$$

The surface integral gives the flux of the quantity through the surface and is a scalar quantity.

Volume integrals are used to sum a quantity over a specified volume. They can be applied to vectors as well as to scalars. For example, the mass within a volume is given by the volume integral of density

$$m = \int_V \rho dV \ . \tag{1.48}$$

Complex Quantities

It is often useful to express a physical quantity as the real part of a complex quantity. This procedure is used extensively when dealing with sinusoids, since the projection on either axis of a uniformly-rotating two-dimensional vector is a sinusoid. A *complex number* may be written either as the sum of a real and an imaginary part or as a magnitude and phase angle, or argument,

$$\underline{A} = A_1 + iA_2 = Ae^{i\theta} \ , \tag{1.49}$$

where the *magnitude*, A, is given by

$$A \equiv \left|A\right| = \sqrt{A_1^2 + A_2^2} \ , \tag{1.50}$$

and the *argument*, θ, by

$$\theta = tan^{-1} \frac{A_2}{A_1} . \qquad (1.51)$$

A_1 is called the *real component* and A_2 the *imaginary* one.

The *complex conjugate* of a complex number has the same amplitude but negative argument

$$\underline{A}^* \equiv A_1 - iA_2 = Ae^{-i\theta} , \qquad (1.52)$$

from which an alternative expression for the amplitude is

$$A = \sqrt{\underline{A} \cdot \underline{A}^*} . \qquad (1.53)$$

When using complex quantities in physical equations, it should be remembered that the physical quantities they represent are their real parts, sometimes written $RP(\underline{A})$. Often the RP is omitted, since it is understood that physical quantities are real. Equations written between complex quantities are also valid equations between their real components. It is useful to think of i as a 90° rotational operator:

$$i \equiv \sqrt{-1} = e^{i(\pi/2)} . \qquad (1.54)$$

From this it also follows that -1, which is i multiplied by itself, represents a rotation of 180°, or π radians.

Interpretation of the real and imaginary parts of a complex number as projections of a vector on the real and imaginary axes leads to several relations between exponential and trigonometric functions. Thus, Eq. 1.49 can be written

$$\underline{A} = A_1 + iA_2 = A \cos\theta + iA \sin\theta = Ae^{i\theta} , \qquad (1.55)$$

from which it follows that

$$e^{i\theta} = \cos\theta + i \sin\theta \qquad (1.56)$$

and that

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos\theta - i \sin\theta . \qquad (1.57)$$

By simultaneous solution of Eqs. 1.56 and 1.57,

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad (1.58)$$

and

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) . \qquad (1.59)$$

Fourier Series

Provided only that linearity can be assumed, any arbitrary function can be represented by an infinite series of functions of a prescribed set. Functions that are useful in mathematical analyses of physical problems include trigonometric functions, Bessel functions and Legendre polynomials. The Fourier method using trigonometric functions is the most popular. Trigonometric functions have the important property that all their derivatives are trigonometric functions, and that every even derivative is the same function.

If $f(x)$ is a continuous function defined in the interval $-L < x < L$, then $f(x)$ can in general be represented by

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{1.60}$$

where the coefficients of the cosine and sine series satisfy the relationships:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 0,1,2,\ldots \tag{1.61}$$

and

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1,2,\ldots. \tag{1.62}$$

The coefficient a_o calculated from Eq. 1.61 with $n = 0$ is twice the average value of $f(x)$ over the interval.

Another useful form of this series makes use of the relations between trigonometric functions and exponentials given in Eqs. 1.58 and 1.59 to replace the sine and cosine terms by a complex exponential,

$$f(x) = \sum_{n=-\infty}^{\infty} \underline{C}_n e^{i(n\pi x/L)}, \tag{1.63}$$

where $\underline{C}_n$ can be expressed in terms of a_n and b_n or calculated directly from

$$\underline{C}_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i(n\pi x/L)} dx \qquad n = 0,1,2,\ldots. \tag{1.64}$$

Normally the number of terms required to represent a given function over a restricted interval is fairly small, ten terms usually being sufficient.

Fourier Integrals and Transforms

To analyze continuous, non-periodic functions, it is necessary to let L approach infinity. The number of important terms increases accordingly, and in the limit the sum becomes replaceable by an integral. The *Fourier integral* expression for a non-periodic function may be written

$$f(x) = \sum_{-\infty}^{\infty} \underline{F}(\eta) e^{i\eta x} \, d\eta \, , \qquad (1.65)$$

where η replaces $n\pi/L$ in Eq. 1.63 as n and L both approach infinity. The function $\underline{F}(\eta)$ is called the *Fourier transform* of $f(x)$ and is given by

$$\underline{F}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, e^{-i\eta x} \, dx \, . \qquad (1.66)$$

Fourier transforms are frequently used in acoustics in transferring from the time domain to spectral space. It is the validity of the Fourier transform concept that makes spectral analysis of complex signals so valuable.

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