ME525 Applied Acoustics Lecture 9, Winter 2022
The acoustic point source (monopole)
The Green's function
Boundary conditions and acoustic doublet

Peter H. Dahl, University of Washington

Continuing with $ka << 1$ and $kL << 1$ limit, monopole source and acoustically compact source

In previous lecture we invoked the $ka << 1$ limit, and with minor rearrangement the pressure is

$$p(r,t) = -i\omega(\rho_0 u_0 4\pi a^2)\frac{e^{ikr}}{4\pi r}e^{-i\omega t} \tag{1}$$

Thus the strength of this acoustic source is defined by the time derivative of *mass flow*, or described another way, it is the *rate of change of mass flow* introduced per unit volume.

After this we defined an *effective source strength* by bundling everything and putting $q = -i\omega(\rho_0 u_0 4\pi a^2)$ giving,

$$p(r,t) = \frac{q}{4\pi r}e^{ikr-i\omega t} \tag{2}$$

where the source is at the center of the coordinate system and pressure is function only of radial coordinate r. So, this source no longer has any length scale a. This length scale has been removed on the assumption that $ka << 1$, and we can replace the sphere of radius a with effective source of strength q.

This defines the concept of concept of a point-like source or *acoustic monopole*. Such a source will generate wave motion in no preferred direction, producing a wave which spreads spherically outward. If there no boundaries, i.e., the medium is infinite in extent, the waveform depend only on the range r from the center of the source, and not depend on the spherical angles α, ϕ as shown in Fig. 1 of Lecture 7.

Furthermore, such a source need not have originally in the form of an exact sphere. The source may instead have some complicated shape (Fig. 1) with characteristic length scale L. We arrive an extraordinarily useful rule: if the characteristic scale L of source is such that $L << \lambda$ where λ is the acoustic wavelength, then the source is *acoustically compact*, and can be viewed as a monopole source. Once the source is deemed acoustically compact the scale L is no longer relevant. The source can be modeling as Eq. (2), where the source strength, q is determined empirically by measurement.

For example, if p_{rms} is measured at range r m from the source, then we can estimate $|q|$ as follows

$$\frac{|q|}{4\pi}\frac{1}{\sqrt{2}}\frac{1}{r} = p_{rms} \tag{3}$$

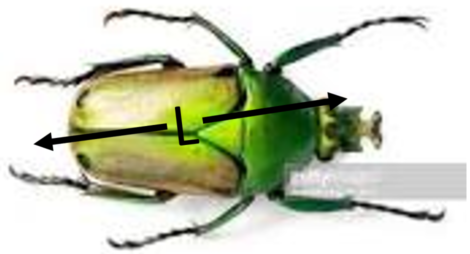

Figure 1: An acoustic source with characteristic scale L for which $kL << 1$ which can be modeled as an *acoustically compact* or source, or monopole.

giving at least a value for $|q|$. Often that is all we are after as the important physics relating to sound propagation is embodied in the factor $\frac{e^{ikr}}{r}$.

Continuing with the Green's function

We next further generalized to find the pressure at a *field point* $\vec{r}$, given a source at an arbitrary *source point* $\vec{r_0}$ that need not be at origin (Fig. 2) as follows:

$$p(\vec{r}, t) = \frac{q}{4\pi|\vec{r} - \vec{r_0}|} e^{ik|\vec{r}-\vec{r_0}|-i\omega t} \tag{4}$$

Equation (4) satisfies the inhomogeneous Helmholtz equation, for which the delta function on the RHS represents a point source of strength q at position $\vec{r_0}$ such that

$$(\nabla^2 + k^2)p = -q\delta(\vec{r} - \vec{r_0}) \tag{5}$$

Further compress notation by defining $R = |\vec{r} - \vec{r_0}|$, such that

$$g = \frac{e^{ikR}}{4\pi R} \tag{6}$$

and call g the *free space* Green's function because g satisifies

$$(\nabla^2 + k^2)g = -\delta(\vec{r} - \vec{r_0}) \tag{7}$$

in an *unbounded* medium.

Note the physical dimension of g is $1/L$. As currently constructed, g embodies all the range-dependent and phase properties of a sound field with point source located at $\vec{r_0}$, but to bring a more useful dimension of pressure, g must be multiplied by a calibration constant. The particular form of g in Eq.(6) which concentrated or "impulse-like" in space is known as a *harmonic* Green's function. In this course we use primarily harmonic Green's function solutions, represent-

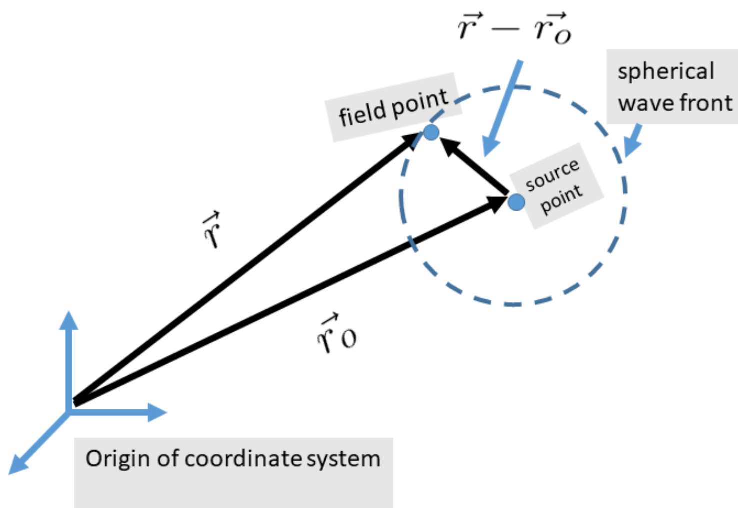

Figure 2: An acoustic source at the source point $\vec{r_0}$ producing the acoustic field at field point $\vec{r}$.

ing a single-frequency, or narrow band condition, and by Fourier superposition we can combine multiple frequencies. A Green's function concentrated in both space and impulsive time is discussed in Pierce (1989), see also Tolstoy (1973).

The Green's function is a model for sound propagation that is proportional to acoustic pressure, differing from pressure only by some multiplicative constant that can complex. The constant may already be known from theory, or determined empirically by measurement. In many applications the constant isn't used as the Green's function usually embodies most if not all the important physics sound propagation.

The expression by way Eq.(5) is common in engineering and physics, where a field quantity, here sound pressure, is governed by a linear partial differential equation with an inhomogeneity at location $\vec{r_0}$ acting as a source term. This pressure field is everywhere smooth and analytic (possesses spatial derivatives that are not infinite), except at the source point it can only be described by a delta function inhomogeneity. This is not unlike a wave created on water, for which the wave field is analytic–except at the point on that surface where the rock splashed (the location of delta function) and produced this wave. In summary: the acoustic field generated by a delta function inhomogeniety is sometimes formally referred to as the *Green's function for the problem.*

Combination of two point sources to satisfy a boundary condition: the acoustic doublet

The Green's function of Eq.(6) is our most basic one which does not need to contend with boundary conditions such as an air-water interface. A simple but realistic boundary condition encountered in acoustics is an acoustic source below an air-water interface. This is called the Lloyd mirror problem and the boundary condition is called ″pressure release″ meaning the acoustic pressure field must vanish on the boundary. Such boundary, again when viewed from source below, is also referred to as an 'acoustically soft' surface. The 'soft' description arises from the characteristic impedance of two acoustic media involved in the problem: water where ρc is of order 1.5 10^6, and air where ρc is $\sim$ 400. In general, sound impinging on a boundary backed by one medium with characteristic impedance much less than the medium supporting the incident field will also behave approximately as a pressure release boundary, or be 'acoustically soft' .

Note that opposite happens for airborne sound over water; in this case the upper medium is supporting the incident field, and with characteristic impedance of the upper air medium being much less than of the lower water medium the boundary can be considered 'acoustically hard'. Later more general boundary conditions are addressed, as say between soft and hard tissue in medical ultrasound, or between sea water and the seabed, which requires a description of the material properties in terms of the characteristic impedance involved in each boundary material. For now, though, so long as an 'acoustically soft' (or 'acoustically hard') approximate description applies, some very combinations of Green's functions can be applied to satisfy the boundary condition.

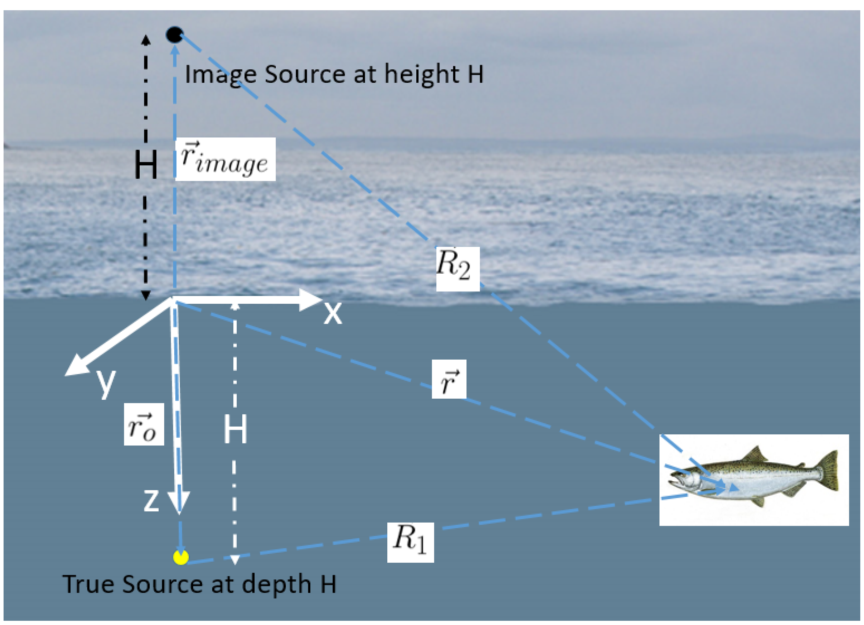

Figure 3: The geometry of the Lloyd Mirror problem showing a true source at depth H and image source at height H.

The pressure release boundary condition is satisfied by combining the true source with an image source (Fig. 3). The image is of opposite sign such that combination of two sources along the boundary equals zero. In this case the Green's function take the following form

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{8}$$

where $R_1 = |\vec{r} - \vec{r_0}|$ and $R_2 = |\vec{r} - \vec{r_{image}}|$. We can set $\vec{r_0} = [0, 0, H]$ and $\vec{r}_{image} = [0, 0, -H]$ in terms of the x, y, z coordinates.

The combination of two free-space Green's functions as in Eq. (8) is known as *doublet*. You should convince yourself that this g equals 0 when evaluated at any x, y with $z = 0$, and remember that g is serving as a surrogate for pressure.

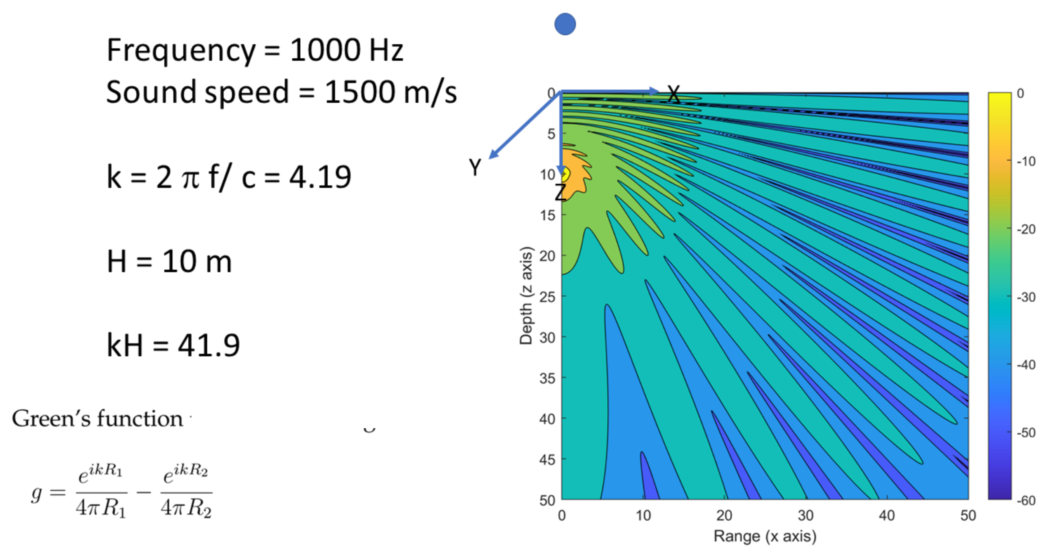

Figure 4: Field in dB with arbitrary reference for acoustic doublet for $H = 10$ and kH = 41.9

Three situations are discussed next for doublet based on differing depths H of the source (Fig. 4-6), and we'll see that a key parameter is kH. In such plots we are interested in how the "strength" of g or pressure, varies with x and z, and so $|g|$ (proportional to pressure amplitude) or $|g|^2$ (proportional to pressure-squared) is plotted. It thus makes sense to continue using decibels, and plot $10 \log_{10} |g|^2$. These three plots display azimuthal symmetry which makes it equally instructive to just plot $20 \log_{10} |g|$ in the x, z plane. Notice what is happening for when the parameter kH is getting smaller.

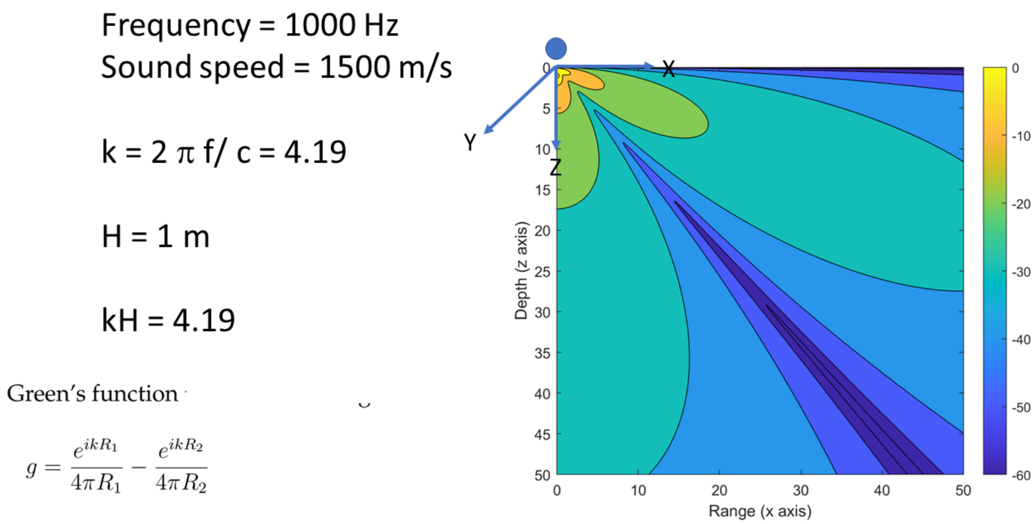

Figure 5: Field in dB with arbitrary reference for acoustic doublet for $H = 1$ and kH = 4.19

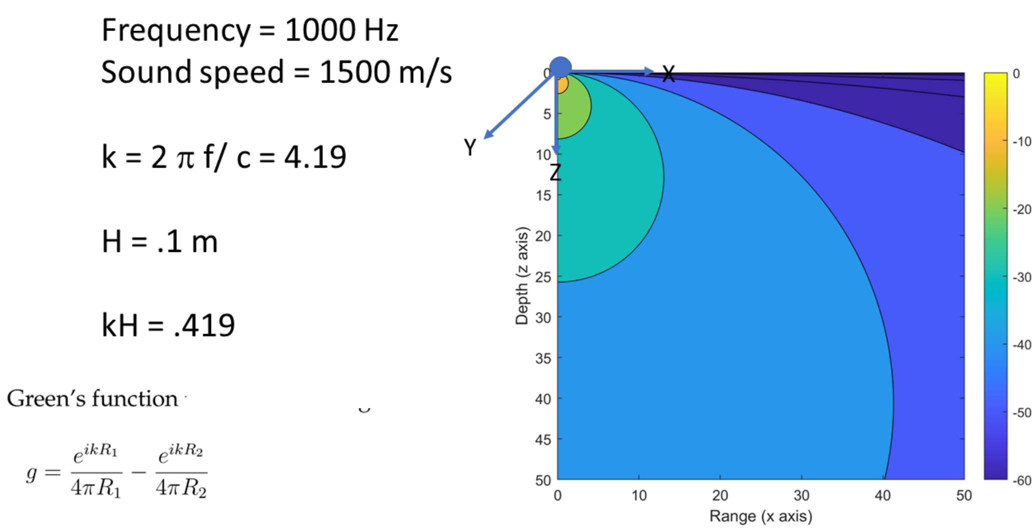

Figure 6: Field in dB with arbitrary reference for acoustic doublet for $H = 0.1$ and kH = 0.419

References

Pierce, A. B, *Acoustics, An Introduction to its Physical Principals and Applications*, (Acoustical Society of America, and American Institute of Physics, 1989)

Frisk, G. V. *Ocean and Seabed Acoustics* (Prentice Hall, Englewood Cliffs, NJ, 1994)

L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, (John Wiley & Sons, New York, 1980)

ME525 Applied Acoustics Lecture 10, Winter 2022
the Green's function
acoustic doublet and dipole

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More on combination of two point sources to satisfy a boundary condition, and method of images

It pays to think about boundary conditions, e.g., between soft and hard tissue in medical ultrasound, or between sea water and the seabed, in terms of the characteristic impedance, $\rho_0 c$, of the acoustic medium on each side of the boundary. This does not tell the whole story, e.g., on one side or both sides there may be layers composed of differing sound speeds and densities, but it gives a good starting approximation. But roughly speaking, for sound in a medium with high $\rho_0 c$ impinging on a boundary to another medium with very low $\rho_0 c$, the boundary condition will behave approximately as a 'pressure release' boundary, meaning the acoustic pressure must equal zero along this boundary. In the study of underwater acoustics, this pressure-release boundary conditions is assumed to be exact. You should examine your self the following ratio: $\frac{(\rho c)_{air}}{(\rho c)_{water}}$ using nominal values for each.

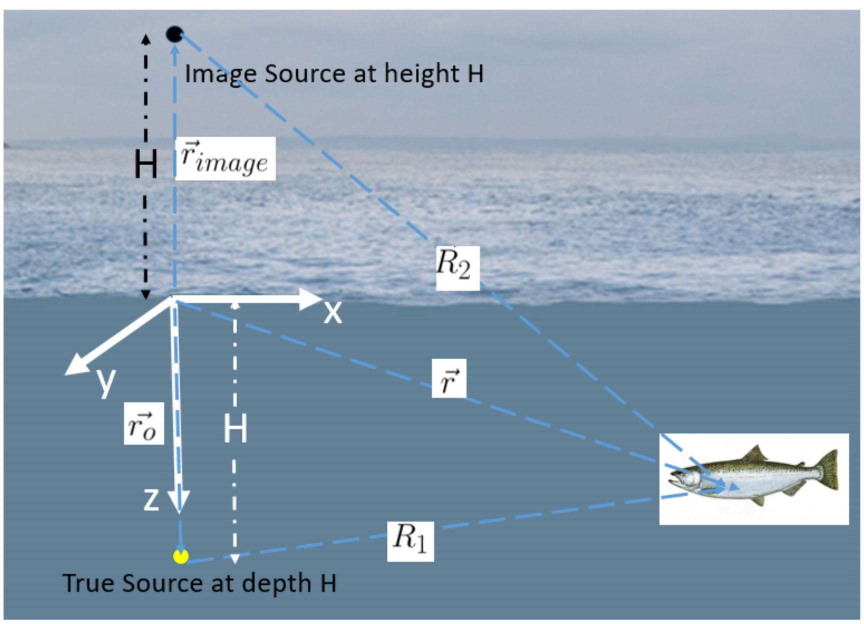

Figure 1: The geometry of the Lloyd Mirror problem showing a true source at depth H and image source at height H.

The pressure release boundary condition is satisfied by combining the true source with an image source (Fig. 1). The image is of opposite sign such that combination of two sources along the boundary equals zero. In this case the Green's function for the Lloyd Mirror problem takes the following form

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{1}$$

where $R_1 = |\vec{r} - \vec{r_0}|$ and $R_2 = |\vec{r} - r_{\vec{image}}|$. We can set $\vec{r_0} = [0, 0, H]$ and $\vec{r}_{image} = [0, 0, -H]$ in terms of the x, y, z coordinates. The method used to find this Green's function is known as the *method of images* (Frisk).

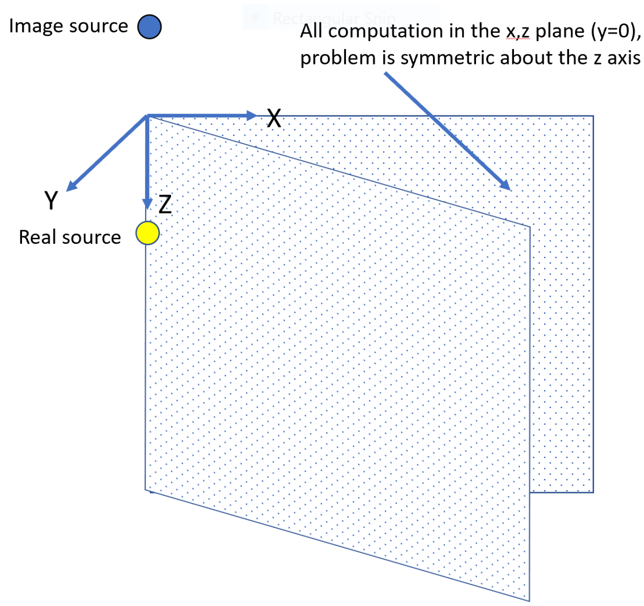

Figure 2: The acoustic field for the geometry in Fig. 1 is symmetric about the z-axis. Thus computing as function of x, z with $y = 0$ is completely sufficient.

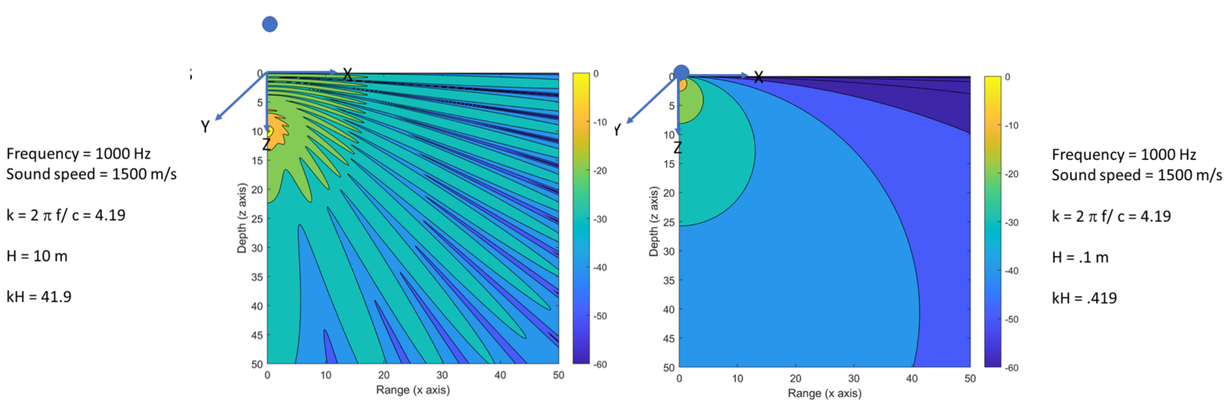

Figure 3: Field in dB with arbitrary reference for two cases with same frequency but differing H as parameterized by $kH >> 1$ and $KH < 1$ acoustic doublet left: $H = 10$ m and kH = 41.9, right: $H = 0.1$ m and kH = 0.419. Similar plots in larger scale are shown in the power point as part of Lecture 9.

Considering the symmetry of the acoustic field about the z axis (Fig. 2), we compute two cases with differing H as parameterized by $kH >> 1$ and $KH < 1$ (Fig. 3). Note that we can also adjust the parameter kH by keeping H the same but adjusting the frequency which changes k. The plots are expressed in terms of contours of $10\log_{10}|g|^2$, or $20\log_{10}|g|$, in the x, z plane, a quantity we might convey to someone else (your co-worker, your research adviser, whomever) in terms of decibels (dB). But the dBs here are obviously not the same as, say, sound pressure level (SPL) in dB reference to 20μPa (air) or 1μPa (underwater). However, considering the fact that g is proportional to acoustic pressure, then there exists some constant-decibel offset to convert results in Fig. 3 to a SPL in dB reference to 1μPa.

We may not be interested in that particular constant-decibel offset, as the more interesting effects are contained in the properties of g as shown here being a function of x, z. Why plot $10\log_{10}|g|^2$ and not just $|g|$ or $|g|^2$? Notice the case $kH >> 1$ with about 14 lobes, or acoustic *beams*, for which $10\log_{10}|g|^2$ varies between about -20 to -30 dB within a beam to about -60 dB outside the beam– let's say a difference of 30 dB or about 1000-fold change in in the value of $|g|^2$. Regions where $|g|^2 \sim -60$ dB are referred to as being in a *null*. We get such a null region because of the interference pattern set up by the (positive) real source and (negative) image source. Though not seen as clearly for case $kH >> 1$, there is very strong null along $z = 0$ boundary, where $|g| = 0$ and any decibel representation would give $-\infty$, a decibel level that is of course not captured in the contour plot. In any case, with such large variation in the field strength it pays to express it in terms of decibel level. (By the way, it is good practice to reserve the word 'level' when talking language involving decibels.)

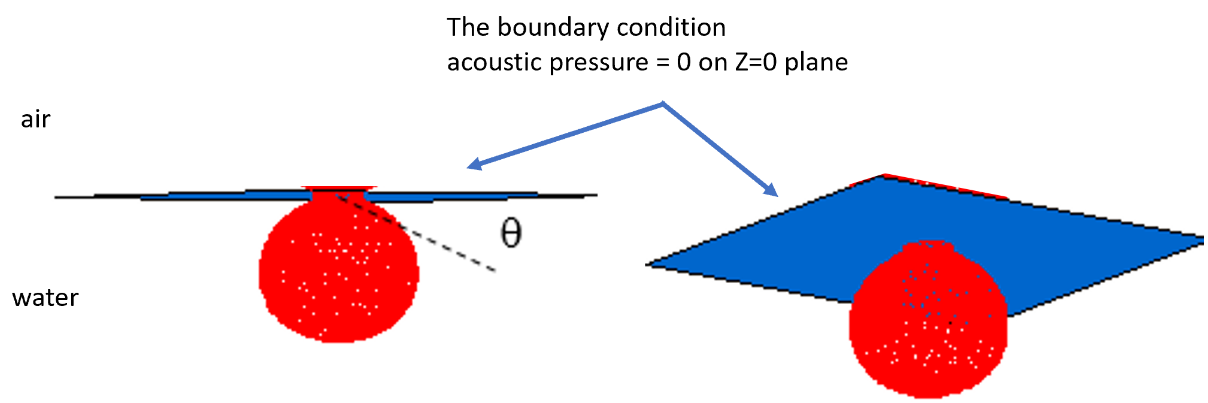

Figure 4: 3D rendition of the very broad beam which results for the acoustic doublet (Fig. 1) when $kH <$.

Turning now to the case $kH < 1$ in Fig. 3, there is also beam and null features, but now the lobes are quite large. For example, take the large -30 dB contour shaded area, and remember this contour swings around 360° about the z axis. It looks something like the rendition in Fig. 4, where one angle with respect to the sea surface, θ, describes everything but dependence on range (an equivalent formulation involves an angle with respect to the z axis).

The acoustic dipole

Now look at the doublet for the case $kH << 1$ (i.e., not too different from Fig. 3 case $kH < 1$) , where $2H$ is the separation between a source and its image which as opposite sign. This is called a *dipole*.

Using the same coordinate system as in Fig. 1, recast R_1 as

$$R_1 = \sqrt{x^2 + y^2 + z^2 - 2zH + H^2} = r\sqrt{1 - 2zH/r^2 + H^2/r^2} \tag{2}$$

then evaluate

$$kR_1 = r\sqrt{k^2 - 2kzH/r^2 + (kH)^2/r^2} \tag{3}$$

In the limit $kH << 1$ the last term is ignored thus $R_1 \approx r(1 - zH/r^2)$, or put $R_1 = r - H\sin\theta$. Similarly we put $R_2 = r + H\sin\theta$. Now, the source $\vec{r_0}$ and image $\vec{r}_{image}$ locations are out of the picture, everything is described by r and θ as in Fig. 5 (the depth coordinate z of the field point included in θ.)

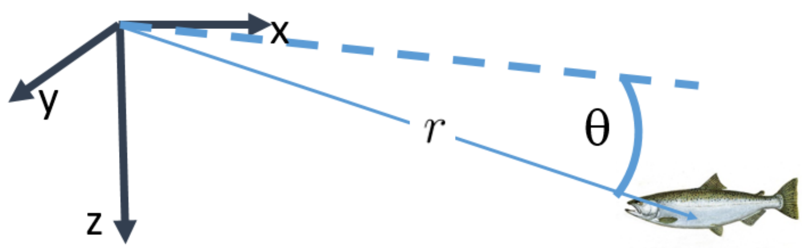

Figure 5: Situation describing a dipole located at center of coordinate system. The acoustic field is function of r and θ.

Now start with the Green's function that satisfies the free-surface boundary condition (discussed previously) composed of source and image of opposite sign (or doublet).

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{4}$$

where $R_1 = |\vec{r} - \vec{r_0}|$ and $R_2 = |\vec{r} - \vec{r_{image}}|$. Now analyze the doublet for the case $kH << 1$, where $2H$ is the separation between a source and its image. This is called a *dipole*.

The approximations for R_1 and R_2, can now be inserted into Eq.(4). In doing so, encounter $e^{\pm ikH\sin\theta}$, which can be approximated as $1 \pm ikH\sin\theta$, consistent with the original $kH << 1$ assumption. This leads to the final result for the dipole Green's function

$$g = 2H\sin\theta\frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}) \tag{5}$$

Observe once again the dimension of this Green's function is $1/L$, as is the case for the two monopoles (source and image) used for the Green's function in Eq.(1). However the dipole now has two terms–one which we be will shown to fade away as $kr >> 1$.

References

Frisk, G. V. *Ocean and Seabed Acoustics* (Prentice Hall, Englewood Cliffs, NJ, 1994)

ME525 Applied Acoustics Lecture 11, Winter 2022
Acoustic Dipole

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The acoustic dipole strength $|f_D|$

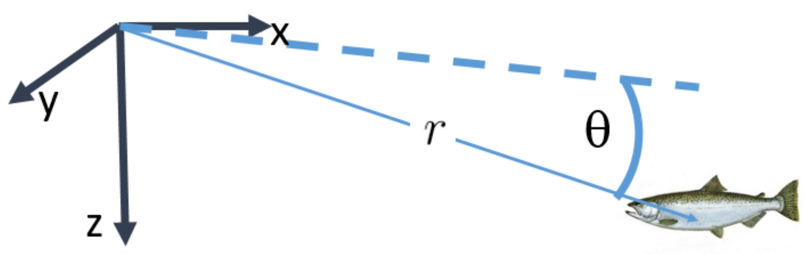

Figure 1: Situation describing a dipole located at center of coordinate system. The acoustic field is function of r and θ.

Let's continue our discussion on the basic dipole Green's function derived in the last lecture.

$$g = 2H \sin\theta \frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}). \tag{1}$$

Recall Eq.(1) originates from the "doublet" consisting of two free-space Green's function sources of opposite sign

$$g = \frac{e^{ikR_1}}{4\pi R_1} - \frac{e^{ikR_2}}{4\pi R_2} \tag{2}$$

where these sources were separated by H and brought closer together such that $kH << 1$. Next incorporate a new "source strength" q to obtain pressure

$$p(r,t) = (q2H) \sin\theta \frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}) \tag{3}$$

where harmonic time dependence $e^{-i\omega t}$ is assumed.

Apply now the same approach used to establish q for point source monopole, instead here allow $2H$ to shrink to 0 while increasing q to keep $(q2H)$ finite. Identify $|f_D|$ as this new quantity to replace $(q2H)$ which is called the *dipole strength*. Thus, we now more formally express the pressure from the dipole as

$$p(r,t) = |f_D| \sin\theta \frac{e^{ikr}}{4\pi r}(-ik + \frac{1}{r}). \tag{4}$$

Note the dimension of $|f_D|$: in MKS it must be N/m, or equivalently Pa-m and the dipole unlike the monopole has two terms: one that dominates in the near field, $1/r$, and that dominates the far field ik.

Vector properties of the dipole

The dipole example just presented is one with the axis oriented perpendicular to the boundary (this axis being the line connecting the real source and and image source, or z-axis as in Fig. 1). This is a very appropriate model, for example, for low frequency ship noise where $kH << 1$ with H being the depth of the noise generating mechanism such as the ship propeller, underwater noise caused by rain drops, and by bubbles bursting near the sea surface due to the action of wave breaking.

This dipole example also demonstrates how the boundary between air and water is satisfied upon either placing a dipole on that boundary, or in the case of the doublet, placing a source a distance H below the boundary and a negative image a distance H above the boundary. But more generally we want to place a dipole anywhere in space to represent a source with properties of two closely-space monopole sources of opposite phase (or sign), i.e. there is no requirement to have the dipole source be associated with a boundary. The dipole can be oriented in any direction and this direction will be the *dipole axis*.

To do this, make the strength $|f_D|$ represent the magnitude of a vector $\vec{f_D}$ called the *dipole moment vector*, where $\vec{f_D}$ is aligned with the dipole axis, and by convention points towards the positive side of the dipole. Thus $\vec{f_D}$ points downward in Fig. 1, or in the case just described where the dipole represents a source very close (in the sense of $kH << 1$) to the air-water interface. To use $\vec{f_D}$ in a more general orientation we need to restore a full vector description $\vec{r}$ for the field point, with the angle α (Fig. 2) given by

$$\cos\alpha = \frac{\vec{f_D} \cdot \vec{r}}{|\vec{f_D}||\vec{r}|} \tag{5}$$

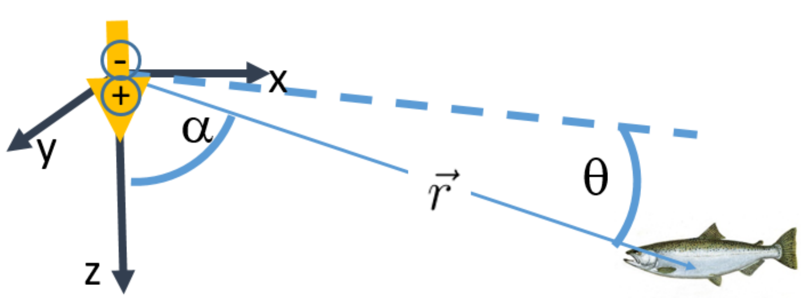

Figure 2: Showing orientation of the dipole moment vector (orange arrow), with positive source ⊕ underwater and negative source ⊖ above. The field point to the fish is now described with vector $\vec{r}$

The dipole moment vector in arbitrary orientation is shown in Fig. 3, and the pressure at the field point (with time dependence $e^{-i\omega t}$) is given by

$$p(FP) = \frac{1}{4\pi}\frac{\vec{f}_D \cdot (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|}\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r} - \vec{r}_0|}(-ik + \frac{1}{|\vec{r} - \vec{r}_0|}) \tag{6}$$

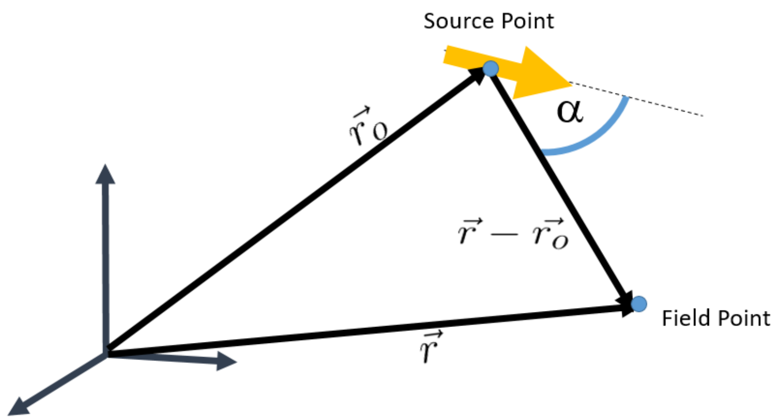

Figure 3: Showing orientation of the dipole moment vector (orange arrow), the dipole source location $\vec{r}_0$ and field point $\vec{r}$ with arbitrary orientation within a coordinate system

Return now to the free space Green's function with harmonic time dependence

$$g = \frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r} - \vec{r}_0|} \tag{7}$$

where a pressure at some field point (FP) is

$$p(FP) = qg \tag{8}$$

Note that gradient of g is ∇g is given by (e.g, see Pierce)

$$\nabla g = \frac{1}{4\pi}\frac{(\vec{r} - \vec{r}_0)}{r}(ik - \frac{1}{r})\frac{e^{ikr}}{r} \tag{9}$$

where $r = |\vec{r} - \vec{r}_0|$. Thus evidently Eq. (6) can also be written in the highly compacted form

$$p(FP) = -\vec{f}_D \cdot \nabla g \tag{10}$$

with Eq.(7) and Eq.(9) now representing our two fundamental source types, with monopole related to g and dipole related to ∇g.

Directivity

Dan Russell and colleagues performed a simple, illustrative experiment to demonstrate the concept of directivity from monopoles, dipoles and quadrapoles (a combination of two dipoles). His experiment (Fig. 4) is as follows: four speakers are on turntable and continuous (harmonic or single frequency) sound is broadcast at fixed frequency = 250 Hz. The RMS pressure is measured by a sound level meter (SLM) as the turntable rotates. Speakers (the square boxes) are arranged to be spatially packed together, separated by distance H, say.

The frequency and H are such that $kH << 1$ for this experimental configuration. Therefore, four speakers when broadcasting with the same phase (the black dots) can be considered a single monopole–see (a) top of Fig. 5. Next, take two of the speakers and reverse the wires (polarity) such that the phase is ”negative”, as in (b) top of Fig. 5. This can be considered a single dipole.

The lower part of Fig. 5 shows some results from the Russell *et al.* study. Directively for the monopole is what we intuitively expect: regardless of the position of the turntable, the SLM gives the same result, and this result, rms pressure plotted in terms of dB, forms a circle. In contrast, the dipole directivity exhibits a deep null in the acoustic response as the turntable passes through 90 and 270°. The two halves of the dipole (in this case the two positive and two negative speakers) exhibit an exact phase cancellation and the acoustic pressure resulting from the coherent sum of four sources should vanish. The Russell *et al.* work also measures the acoustic field from a quadrapole (see (d) in upper part of Fig. 5). This will exhibit a kind of clover-leaf directivity pattern.

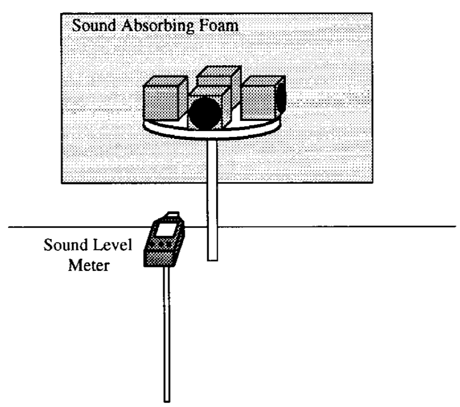

Figure 4: Apparatus for measuring directivity of monopoles, dipoles, and quadrapoles. This is Fig. 5 from Russell *et al.*, ”Monopoles, Dipoles and Quadrapoles: An experiment revisited”, Am. Journal of Physics, 1999.

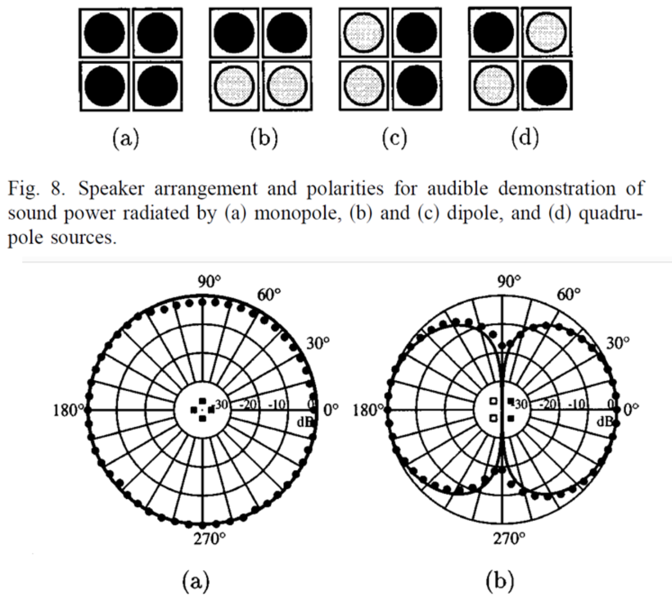

Figure 5: Measurements of directivity pattern for (a) monopole and (b) dipole. This is portion of Fig. 8 from Russell *et al.*, "Monopoles, Dipoles and Quadrapoles: An experiment revisited", Am. Journal of Physics, 1999.

Arrangement of 4 speakers, separated by H and driven by a frequency such that $kH << 1$ amounts to an idealized directivity problem where measurement well predicted by simple theory. More often for realistic noise emission problems, an empirical measurement is required as in the case of directivity of jet noise measured on the ground (Fig. 6).

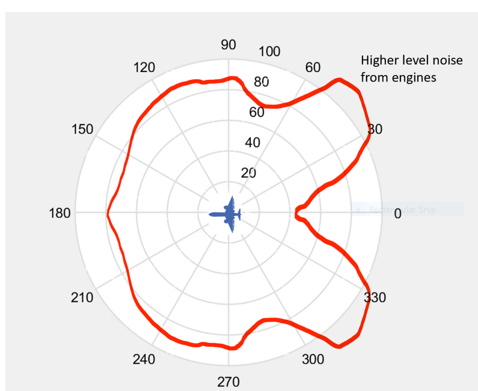

Figure 6: Notional directively of jet noise emission over broad range of frequencies as function of angle.

References

Pierce, A. B, *Acoustics, An Introduction to its Physical Principals and Applications*, (Acoustical Society of America, and American Institute of Physics, 1989)