Ocean Acoustic Modeling in MATLAB

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- Rays
- Modes
- Wavenumber integration (FFP)
- Parabolic Equation

Governing equation (the wave equation):

$$\nabla^2 p + \frac{1}{c^2(r,z)} p_{tt} = \frac{-\delta(r-r_s)\delta(z-z_s)}{r}$$

Helmholtz Equation:

$$\nabla^2 p + \frac{\omega^2}{c^2(\mathbf{x})}\, p = -\delta(\mathbf{x}-\mathbf{x}_s)\ ,$$

Ray Theory

Seek a solution of the Helmholtz equation in the following form:

$$p(\mathbf{x}) = e^{i\omega\,\tau(\mathbf{x})} \sum_{j=0}^{\infty} \frac{A_j(\mathbf{x})}{(i\omega)^j}$$

where, $k = \omega/c_0$.

- called the *Ray Series*.
- generally divergent.
- provides an *asymptotic* approximation

Differentiate the ray series ...

$$p_x = e^{i\omega\tau} \left[i\omega\tau_x \sum_{j=0}^{\infty} \frac{A_j}{(i\omega)^j} + \sum_{j=0}^{\infty} \frac{A_{j,x}}{(i\omega)^j} \right] ,$$

and

$$p_{xx} = e^{i\omega\tau} \left\{ \left[-\omega^2 (\tau_x)^2 + i\omega\tau_{xx} \right] \sum_{j=0}^{\infty} \frac{A_j}{(i\omega)^j} + 2i\omega\tau_x \sum_{j=0}^{\infty} \frac{A_{j,x}}{(i\omega)^j} + \sum_{j=0}^{\infty} \frac{A_{j,xx}}{(i\omega)^j} \right\} .$$

Substitute back in the Helmholtz equation ...

$$\nabla^2 p = e^{i\omega\tau}\left\{\left[-\omega^2|\nabla\tau|^2 + i\omega\nabla^2\tau\right]\sum_{j=0}^{\infty}\frac{A_j}{(i\omega)^j} + 2i\omega\nabla\tau\cdot\sum_{j=0}^{\infty}\frac{\nabla A_j}{(i\omega)^j} + \sum_{j=0}^{\infty}\frac{\nabla^2 A_j}{(i\omega)^j}\right\}.$$

Equate terms of like order in ω

$$\begin{array}{lrcll}
O(\omega^2) \quad : & |\nabla\tau|^2 & = & c^{-2}(\mathbf{x}) & \quad \text{Eikonal} \\
O(\omega) \quad : & 2\nabla\tau\cdot\nabla A_0 + (\nabla^2\tau)A_0 & = & 0 \quad \text{Transport} & \\
O(\omega^{1-j}) : & 2\nabla\tau\cdot\nabla A_j + (\nabla^2\tau)A_j & = & -\nabla^2 A_{j-1}\,, & \qquad j = 1, 2, \ldots
\end{array}$$

Solving the Eikonal Equation (Method of Characteristics)

Define *rays* as curves perpendicular to the *wavefronts* of $\tau(\mathbf{x})$:

$$\frac{d\mathbf{x}}{ds} = c\,\nabla\tau$$

But, phase is still unknown.

Lots of work ...

$$\frac{d}{ds}\left(\frac{1}{c}\,\frac{d\mathbf{x}}{ds}\right) = -\frac{1}{c^2}\,\nabla c$$

Rays are now defined just in terms of $c(\mathbf{x})$!

Ray equations in cylindrical coordinates

$$\begin{aligned} \frac{dr}{ds} &= c\,\xi(s)\,, \qquad & \frac{d\xi}{ds} = -\frac{1}{c^2}\frac{dc}{dr}\,, \\ \frac{dz}{ds} &= c\,\zeta(s)\,, \qquad & \frac{d\zeta}{ds} = -\frac{1}{c^2}\frac{dc}{dz} \end{aligned}$$

$[r(s), z(s)]$ is the trajectory of the ray.

Initial Conditions

$$\begin{aligned} r &= r_s\,, \qquad & \xi = \frac{\cos\theta}{c(0)}\,, \\ z &= z_s\,, \qquad & \zeta = \frac{\sin\theta}{c(0)} \end{aligned}$$

Solving the eikonal equation

Eikonal equation:

$$\nabla\tau \cdot \nabla\tau = \frac{1}{c^2}\,.$$

Therefore

$$\nabla\tau \cdot \frac{1}{c}\frac{d\mathbf{x}}{ds} = \frac{1}{c^2}\,,$$

or

$$\frac{d\tau}{ds} = \frac{1}{c}\,.$$

(eikonal equation in ray coordinate s). Original nonlinear PDE is now a linear ODE

Solution:

$$\tau(s) = \tau(0) + \int_0^s \frac{1}{c(s')}\,ds'\,.$$

Phase of the wave is delayed by its travel time.

NORMAL MODES

Helmholtz equation (2-D):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial p}{\partial r}\right)+\frac{\partial^2 p}{\partial z^2}+\frac{\omega^2}{c^2(z)}\,p=-\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}$$

Solve by *separation of variables*. Seek $p(r,z)=\Phi(r)\Psi(z)$.

Depth-separated equation

$\Psi(z)$ must satisfy:

$$\begin{aligned} \frac{d^2\Psi_m(z)}{dz^2} + \left[\frac{\omega^2}{c^2(z)} - k_{rm}^2\right]\Psi_m(z) &= 0 \\ \Psi(0) &= 0 \\ \left.\frac{d\Psi}{dz}\right|_{z=D} &= 0 \end{aligned}$$

This must be solved *numerically* for k_m and $\Psi_m(z)$ (the eigenvalues and eigenfunctions of this Sturm-Liouville problem).

- There are an infinite number of modes!
- Fortunately, we can make do with a finite number
- They can be scaled arbitrarily

Depth-separated equation continued ...

Orthogonality property:

$$\int_0^D \Psi_m(z)\,\Psi_n(z)\,dz = 0\,, \qquad \text{for} \qquad m \neq n$$

Normalization:

$$\int_0^D \Psi_m^2(z)\,dz = 1$$

Range-separated equation

$\Phi(r)$ must satisfy:

$$\frac{1}{r}\frac{d}{dr}\left[r\,\frac{d\Phi_n(r)}{dr}\right] + k_n^2\,\Phi_n(r) = 0\,.$$

The range functions are easy:

$$\Phi_n(r) = H_0^{(1)}(k_n r).$$

We can evaluate the Hankel functions exactly but if we're more than λ away, we use:

$$H_0^{(1)}(kr) \to \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\frac{\pi}{4})}$$

A summing up ...

$$p(r,z) = \frac{i}{4}\sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,H_0^{(1)}(k_m r)$$

or, using the asymptotic approximation to the Hankel function,

$$p(r,z) \simeq \frac{i}{\sqrt{8\pi r}}\,e^{-i\pi/4}\sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_m r}}{\sqrt{k_m}}$$

Transmission Loss

$$\mathrm{TL}(r,z) \simeq -20\log\left|\sqrt{\frac{2\pi}{r}}\sum_{m=1}^{\infty} \Psi_m(z_s)\,\Psi_m(z)\,\frac{e^{ik_m r}}{\sqrt{k_m}}\right|\ .$$

Example: Isovelocity profile (c(z) = 1500 m/s)

- Solve for the modes

$$\frac{d^2\Psi_m(z)}{dz^2} + \left[\frac{\omega^2}{c^2} - k_m^2\right]\Psi_m(z) = 0$$
$$\Psi(0) = 0$$
$$\left.\frac{d\Psi}{dz}\right|_{z=D} = 0$$

General solution:

$$\Psi_m(z) = A\sin(k_z z) + B\cos(k_z z)\ ,$$

where the vertical wavenumber k_z is:

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - k^2}\ .$$

- Top BC implies $B = 0$.
- Bottom BC implies

$$Ak_z\ \cos(k_z D) = 0\ ,$$

requiring

$$k_z D = \left(m - \frac{1}{2}\right)\pi\,, \qquad m = 1, 2, \ldots\,,$$

Thus,

$$k_m = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left[\left(m - \frac{1}{2}\right)\frac{\pi}{D}\right]^2}\,, \qquad m = 1, 2, \ldots$$

- Corresponding eigenfunctions are given by

$$\Psi_m(z) = \sqrt{\frac{2}{D}}\,\sin(k_{zm} z)$$

- Sum up the modes to get the pressure field

$$p(r, z) = \frac{i}{2D}\sum_{m=1}^{\infty} \sin(k_{zm} z_s)\sin(k_{zm} z)\, H_0^{(1)}(k_m r)$$

Numerics

Recall the modal equation:

$$\begin{aligned}
\Psi''(z) + \left[\frac{\omega^2}{c^2(z)} - k^2\right]\Psi(z) &= 0\,, \\
\Psi(0) &= 0\,, \\
\Psi(D) &= 0\,.
\end{aligned}$$

Finite-difference approximation ($\psi_j = \Psi(z_j)$):

$$\psi_j'' = \frac{\psi_{j-1} - 2\psi_j + \psi_{j+1}}{h^2} + O(h^2)\,.$$

$$\begin{aligned}
\frac{\psi_{j-1}}{h^2} + \left\{\frac{-2}{h^2} + \frac{\omega^2}{c^2(z_j)} - k^2\right\}\psi_j + \frac{\psi_{j+1}}{h^2} &= 0\,, \qquad j = 1, \ldots N-1\,, \\
\psi_0 &= 0 \\
\psi_{N+1} &= 0
\end{aligned}$$

Matrix form

$$\left[\mathbf{A} - k^2\mathbf{I}\right]\,\psi = 0\,.$$

This equation has N eigenvalues $k^{(m)}$ and corresponding eigenvectors $\psi^{(\mathbf{m})}$.

$\psi^{\mathbf{m}}$ is the vector with components $\Psi_1^m, \Psi_2^m, \ldots \Psi_N^m$. Thus, $\psi_j^{(m)} \simeq \Psi_m(z_j)$.

$$\mathbf{A} = \begin{bmatrix} d_1 & e_2 & & & & & \\ e_2 & d_3 & e_3 & & & & \\ & e_3 & d_2 & e_4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & e_{N-2} & d_{N-2} & e_{N-1} & \\ & & & & e_{N-1} & d_{N-1} & e_N \\ & & & & & e_N & d_N \end{bmatrix}$$

where,

$$d_j = \frac{-2}{h^2} + \frac{\omega^2}{c^2(z_j)}, \qquad e_j = \frac{1}{h^2}$$

Mode Normalization

The integral term can be evaluated by the trapezoidal rule. That is,

$$\int_0^D \Psi_m^2(z)\,dz \simeq h\left(\frac{1}{2}\phi_0 + \phi_1 + \phi_2 + \cdots + \phi_{N-1} + \frac{1}{2}\phi_N\right) ,$$

where

$$\phi_j = \left(\psi_j^{(m)}\right)^2 .$$

Say it all in matrices:

1. Solve the algebraic eigenvalue problem $\mathbf{A}\psi = \lambda\psi$ ($\mathbf{A}$ defined above).
   Result: $\psi^{(\mathbf{m})}, \lambda^{(m)}, m = 1, \ldots, N$.
   Note: $k_m \leftarrow \sqrt{\lambda}$.
2. Sort k_m and keep the modes with the largest real part.
3. Normalize the eigenfunctions

$$\psi^{(m)} \leftarrow \frac{\psi^{(m)}}{\|\psi^{(m)}\|}$$

4. Assemble the eigenvectors into a matrix

$$\psi = \left[\begin{array}{cccc} \psi^{(\mathbf{1})} & \psi^{(\mathbf{2})} & \ldots & \psi^{(\mathbf{M})} \end{array} \right]$$

   (ψ_{im} is the ith element of the mth mode.)

5. Calculate mode excitation coefficients $\mathbf{C}$:

$$\mathbf{C} \leftarrow \begin{bmatrix} \psi_{isd,1} \\ \psi_{isd,2} \\ \vdots \\ \psi_{isd,M} \end{bmatrix}$$

6. Calculate $\tilde{\psi}$, a mode matrix scaled by the mode excitation:

$$\tilde{\psi} \leftarrow \psi \begin{bmatrix} 1/C_1 & & \\ & \ddots & \\ & & 1/C_M \end{bmatrix}$$

7. Calculate the $\mathbf{\Phi}$ is a phase matrix:

$$\mathbf{\Phi} \leftarrow \begin{bmatrix} 1/\sqrt{k_1} & & \\ & \ddots & \\ & & 1/\sqrt{k_M} \end{bmatrix} \begin{bmatrix} e^{ik_1 r_1} & e^{ik_1 r_2} & \dots & \dots & e^{ik_1 r_{NR}} \\ e^{ik_2 r_1} & e^{ik_2 r_2} & & & e^{ik_2 r_{NR}} \\ \vdots & \vdots & & & \vdots \\ e^{ik_M r_1} & e^{ik_M r_2} & \dots & \dots & e^{ik_M r_{NR}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{r_1} & & \\ & \ddots & \\ & & 1/\sqrt{r_{NR}} \end{bmatrix}$$

8. Sum up the modes:

$$p = \tilde{\psi}\mathbf{\Phi}$$

Recall,

$$p(r,z) \simeq \frac{i}{\sqrt{8\pi r}}\, e^{-i\pi/4} \sum_{m=1}^{\infty} \Psi_m(z_s)\, \Psi_m(z)\, \frac{e^{ik_m r}}{\sqrt{k_m}}$$

where,

Adiabatic Modes

As the modes propagate in a range-dependent environment, they continually change shape.

As they change shape, energy is continously re-distributed amongst the local modes.

However, for sufficiently slowly-varying environments the energy mostly stays in the same mode.

Then, the acoustic field can easily be computed using the *adiabatic mode formula*:

$$p(r,z) \simeq \frac{i}{\sqrt{8\pi r}}\, e^{-i\pi/4} \sum_{m=1}^{\infty} \Psi_m(z_s)\, \Psi_m(r,z)\, \frac{e^{i\int_0^r k_m(r')\,dr'}}{\sqrt{k_m(r)}}\,.$$

FAST FIELD PROGRAM

Apply a Fourier-Bessel transform to the Helmholtz equation:

$$g(k,z) = \int_0^\infty p(r,z) J_0(kr) r dr$$

$$\begin{aligned} \Rightarrow \frac{d^2 g}{dz^2} + \left(\frac{\omega^2}{c^2(z)} - k^2 \right) g &= \delta(z - z_s), \\ g(0) = 0, \qquad \frac{dg}{dz}(D) &= 0 \end{aligned}$$

The pressure field is reconstructed using the inverse transform:

$$p(r,z) = \int_0^\infty g(k,z) J_0(kr) k dk$$

which can be efficiently evaluated using an FFT (Marsh, DiNapoli(1967)):

$$p(r,z) \approx \frac{e^{i\pi/4}}{\sqrt{2\pi r}} \int_0^{K_{max}} g(k,z) \sqrt{k} e^{ikr} dk.$$

Matrix form

$$\left[\mathbf{A} - k^2\mathbf{I}\right] \mathbf{g} = \delta \, .$$

$$\mathbf{A} = \begin{bmatrix} d_0 & e_1 & & & & & \\ e_1 & d_1 & e_2 & & & & \\ & e_2 & d_2 & e_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & e_{N-2} & d_{N-2} & e_{N-1} & \\ & & & & e_{N-1} & d_{N-1} & e_N \\ & & & & & e_N & d_N \end{bmatrix}$$

where,

$$\begin{aligned} d_j &= \frac{-2}{h^2} + \frac{\omega^2}{c^2(z_j)} \\ e_j &= \frac{1}{h^2} \end{aligned}$$

- This is a linear system of N equations.
- δ is a function that is 1 at the source depth and 0 elsewhere.
- For each choice of the horizontal wavenumber k we get a response $g(k, z)$.

Overview of the numerical procedure

1. Set up the matrix **A**.
2. Define a sequence of wavenumbers

$$\begin{aligned}\{k_j &= k_{min} + j\Delta k + i\alpha, j = 1, NK\} \\ \Delta k &= \frac{k_{max} - k_{min}}{NK - 1} \\ \alpha &= \Delta k\end{aligned}$$

3. Solve $[A - k_j I]g = \Delta$ for each wavenumber k_j.
   (Result is a matrix $\mathbf{g}$ with $g_{ij} \simeq g(z_i; k_j)$)
4. Multiply by $\sqrt{k}$:

$$\mathbf{g^T} \leftarrow \mathbf{g} \begin{bmatrix} 1/\sqrt{k_1} & & \\ & \ddots & \\ & & 1/\sqrt{k_{NK}} \end{bmatrix}$$

5. Do an FFT: $g(r, z) = FFT(g(k, z))$

6. Put in the cylindrical spreading and damping factor.

$$p = \mathbf{g} \begin{bmatrix} e^{\alpha r_1}/\sqrt{r_1} & & \\ & \ddots & \\ & & e^{\alpha r_{NR}}/\sqrt{r_{NR}} \end{bmatrix}$$

PARABOLIC EQUATION MODELING

(following Tappert and Hardin (1973))

Recall, Helmholtz equation:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0,$$

Seek,

$$p(r,z) = \psi(r,z) H_0^{(1)}(k_0 r),$$

where $H_0^{(1)}(k_0 r)$ is the Hankel function and satisfies:

$$\frac{\partial^2 H_0^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial H_0^{(1)}}{\partial r} + k_0^2 H_0^{(1)} = 0,$$

We find,

$$\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{2}{H_0^{(1)}}\frac{\partial H_0^{(1)}}{\partial r} + \frac{1}{r}\right)\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0.$$

PE derivation continued ...

Far-field approximation to Hankel function:

$$\frac{\partial^2\psi}{\partial r^2} + 2ik_0\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + k_0^2(n^2-1)\psi = 0.$$

(still elliptic and not marchable)

Small-angle approximation (this is not well-motivated yet):

$$\frac{\partial^2\psi}{\partial r^2} \ll 2ik_0\frac{\partial\psi}{\partial r}.$$

Gives the *standard parabolic equation*

$$2ik_0\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + k_0^2(n^2-1)\psi = 0,$$

Starting Fields

We can't start the PE with a delta function because it radiates uniformly in angle and the PE abuses the high-angle paths.

Choices:

- Gaussian starter (just a smoothed delta function)

$$\psi(0,z) = \sqrt{k_0}\, e^{-\frac{k_0^2}{2}(z-z_s)^2},$$

- Modal starter
- Greene's source

$$\psi(0,z) = \sqrt{k_0}\left[1.4467 - 0.4201 k_0^2 (z-z_s)^2\right] e^{-\frac{k_0^2(z-z_s)^2}{3.0512}},$$

Numerics

Let $\psi_j^i = \psi(r_i, z_j)$, i.e. ψ^i is a vector that approximates the field at range r_i.

$$\frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi$$

is approximately $\mathbf{A}\psi$ where

$$\mathbf{A} = \begin{bmatrix} d_1 & e_2 & & & & & \\ e_2 & d_3 & e_3 & & & & \\ & e_3 & d_3 & e_4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & e_{N-2} & d_{N-2} & e_{N-1} & \\ & & & & e_{N-1} & d_{N-1} & e_N \\ & & & & & e_N & d_N \end{bmatrix}$$

and,

$$\begin{aligned} d_j &= \frac{-2}{h^2} + k_0^2(n^2(z_j) - 1) \\ e_j &= \frac{1}{h^2} \end{aligned}$$

Numerics (continued)

Recall,

$$2ik_0\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}+k_0^2(n^2-1)\psi=0,$$

Thus,

$$2ik_0\frac{\psi^{i+1}-\psi^i}{\Delta r}+A\frac{\psi^{i+1}+\psi^i}{2}=0$$

Rearrange,

$$\left[\frac{2ik_0}{\Delta r}+\frac{\mathbf{A}}{2}\right]\psi^{i+1}=\left[\frac{2ik_0}{\Delta r}-\frac{\mathbf{A}}{2}\right]\psi^i$$

Or,

$$\mathbf{C}\psi^{i+1}=\mathbf{B}\psi^i$$