Computational Ocean Acoustics

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Problems and Exercises

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Preface

Computational Ocean Acoustics is intended both as a tool for the practicing researcher and as a textbook for graduate and senior undergraduate students specializing in underwater acoustics. However, several of our colleagues have pointed out that as a textbook a key component is missing – a problem set. This issue was discussed among the authors at an early stage, but it was decided to publish the book without a problem set. Instead we decided to make a separate booklet with problems and exercises that would be made available at no or very limited cost to instructors of courses in ocean acoustics.

The most important purpose of the problem set is to aid the student in understanding the fundamental concepts described in the textbook. As a new book, *Computational Ocean Acoustics* had not yet been used for teaching purposes at the time it went into print, and a consistent and complete problem set suitable for being "frozen" into the book did not exist. A good problem set develops over time, adjusting to the needs of students and further clarifying material not covered in detail in the body of the text.

At this time, *Computational Ocean Acoustics* has been used as a textbook in several courses given by the authors. As a result, a rather complete, although not perfect, problem set has been developed, suitable for publication as the first edition of *Computational Ocean Acoustics – Problems and Exercises.* By publishing the problem set as a separate booklet, we allow for future modifications reflecting the experience of the authors and others using the book for teaching.

The present problem set mainly contains problems and exercises focusing on the fundamental mathematical and physical concepts of ocean acoustics. Some problems concern the basic numerical aspects associated with the various numerical techniques. Even though several problems require computer coding, there are no problems directly requiring the students to build full scale propagation models. The development of such working codes requires the integration of all the theoretical and numerical concepts covered in the book.

Therefore, it is highly recommended that the traditional homework problems, such as the ones in this volume, are supplemented with "hands on" projects involving direct model development. The student can then learn a great deal by addressing fundamental problems of a physical nature, similar to the ones covered by the numerical examples in the book. The recipes provided at the end of Chapters 3–6 should provide a suitable guide for developing such models.

We would like to stress once again that the present problem set is not a final product, but is expected to develop significantly in the future. A crucial component of such an improvement process is the feedback from instructors and students using it. Therefore, we would highly appreciate receiving any comments and suggestions for improvement, preferably by e-mail to *coa@keel.mit.edu.*

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Chapter 1

Fundamentals of Ocean Acoustics

1.1. In a deep ocean with constant salinity 3.5% the water temperature distribution is assumed to be exponential in depth,

$$T = T_0 \exp(-z/500) ,$$

where T and T_0 are in degrees Celsius, and z is in meters.

a. What is the minimum surface water temperature for which a deep sound channel (SOFAR) will exist?

b. Determine the depth of the channel axis and the associated sound speed as function of the surface temperature.

1.2. Consider a 4000 m deep ocean with constant salinity 3.5%. The water temperature distribution is assumed to be exponential in depth,

$$T = 10 \exp(-z/500) ,$$

where T is in degrees Celsius, and z is in meters.

a. At approximately which latitude would you expect to find such an environment?

b. For a source at 100 m depth, discuss the existence of the various ray paths (RR, RSR, RBR and SRBR) in this environment.

c. What is the surface temperature for which no pure RSR and RBR paths exist?

1.3. In air acoustics, the conventional reference for decibels is dB re 0.0002 dyn/cm^2 as opposed to dB re 1μPa used in ocean acoustics.

a. A human whisper and shout have acoustics powers of about 10^{-10} and 10^{-5} watts, respectively. Express their dB levels using both conventions. What would be the dB levels if the whole world shouted at once (in the same place)? Compare that to a jet or rocket in air or various types of ships in water.

b. If a rock band played at the pain threshold, of about 140 dB, what is its power output in watts. What is its corresponding sound pressure level in water?

c. For a 120 dB source in water (measured one meter from the source), what would its dB level be at ranges 1, 10 and 100 km assuming spherical spreading; cylindrical spreading? The loudest whales have source levels of about 190 dB. Compare this to a rock band.

1.4. An omnidirectional source of frequency f is placed at a distance z_s from an infinitely rigid wall bounding a fluid halfspace with constant sound speed c.

a. Describe the radiation pattern in the limit of $z_s \to 0$.

b. Derive the expression for the number of *Lloyd-mirror* beams.

c. Derive the asymptotic field decay parallel to the wall, and compare to the corresponding pressure-release surface result.

1.5. Estimate the convergence zone (CZ) separation for an Arctic environment with the sound speed profile given below. Assume linear sound speed variation between the profile depths.

Depth (m)	Sound speed (m/s)
0	1438.0
300	1460.0
4000	1519.2

1.6. Write a program for computing and displaying the magnitude and phase of the reflection and transmission coefficients for the interface separating two fluid halfspaces.

a. Use your code to illustrate the concept of a *critical angle* by properly choosing the sound speeds and densities.

b. For grazing angles of incidence smaller and larger than critical, discuss the depth-dependence (direction perpendicular to the interface) of the reflected and transmitted fields.

c. Discuss the behavior of the phase of the reflection coefficient for incident grazing angles less than and larger than critical.

d. Create an example illustrating the concept of an *intromission angle.*

1.7. Derive the expression for the reflection coefficient for a fluid layer overlying an infinitely rigid halfspace. Give a physical explanation for the frequency and grazing angle dependence of the magnitude and phase.

Chapter 2

Wave Propagation Theory

2.1. Sound propagating in a moving medium is governed by a so-called *convected wave equation*. Consider the case where the background flow velocity is uniform in the x-direction with velocity V.

a. Following the procedure in Sec. 2.1, derive the convected wave equation for sound in a one-dimensional environment with flow velocity V:

$$\left(1 - \frac{V^2}{c^2}\right) p_{xx} - \frac{2V}{c^2} p_{xt} - \frac{1}{c^2} p_{tt} = 0 \, .$$

Note that setting $V = 0$ gives the usual wave equation.

b. Show that this equation can also be derived from the standard wave equation by changing to a moving coordinate system $(\xi, \tau) = (x + Vt, t)$.

c. What is the form of this equation in three dimensions?

2.2. Assume an acoustic source is designed as a small, spherical balloon of radius a, within which the pressure is oscillating with frequency f, with maximum pressure amplitude P.

a. Derive the expression for the acoustic pressure vs range.

b. Determine the expression for P which directly yields transmission loss, i.e., unit pressure at $r = 1\,\mathrm{m}$.

2.3. Derive *Green's theorem* for a fluid medium with variable density, where the wave equation is of the form given in Eq. (2.13).

2.4. Make a computer code for computing the magnitude and phase of the plane-wave reflection coefficient at an interface separating two fluid halfspaces.

a. As a test of your code reproduce the results of Figs. 2.10 and 2.11.

b. Discuss in physical terms the grazing angle dependence of the results.

c. Add a second fluid layer in the bottom and then add frequency as an independent variable to your computer program. Contour your reflection results as a function of angle and frequency. Discuss the resulting structure of the contoured output.

2.5. For an ideal waveguide bounded above by a pressure-release surface and below by an infinitely rigid wall, derive a ray expansion for the acoustic field.

2.6. Write a code evaluating the ray expansion in Eq. (2.136) for the pressure field in an ideal waveguide with pressure-release boundaries.

   a. For a 100 m deep waveguide, compute the transmission loss for both source and receiver at depth 36 m, at every 100 m range out to 2 km. Compare your results to Fig. 2.23(b).
   b. Perform a convergence analysis for a few selected ranges and discuss the range dependence.

2.7. Show that Eq. (2.147) represents the sum of the residues of the wavenumber kernel in Eq. (2.143).

2.8. Consider an isovelocity waveguide of thickness D, bounded above and below by infinitely rigid walls.

   a. Derive the characteristic equation for the horizontal wavenumber of the normal modes.
   b. Sketch the vertical pressure distribution of the first few normal modes.
   c. Derive the dispersion relation for the normal modes. Discuss the differences compared to the waveguide with pressure release boundaries.

2.9. Consider an environment similar to the Pekeris waveguide in Fig. 2.25, but with the bottom speed being changed to $c_2 = 1300\,\text{m/s}$.

   a. Make a sketch of the complex wavenumber plane for this problem (similar to Fig. 2.26), indicating the integration contour and the EJP branch cuts.
   b. Discuss the existence of normal modes in this case. If they exist, show their approximate positions.
   c. Make a sketch of the branch cuts corresponding to the vertical wavenumber being purely imaginary, with the corresponding closed integration contour.

2.10. Consider a Pekeris waveguide with the speed of sound $c_1 = 1500\,\text{m/s}$ and density $\rho_1 = 1000\,\text{kg/m}^3$ in the water column, and with $c_2 = 1800\,\text{m/s}$ and $\rho_2 = 2000\,\text{kg/m}^3$ in the bottom. The water depth is 100 m. A line source at depth z_s is generating a plane acoustic field in the waveguide.

   a. Defining the slowness of the mth normal mode as

$$p_m = \frac{k_{xm}}{\omega} \, ,$$

where k_{xm} is the horizontal wavenumber of the mode, state the upper and lower limit of p_m for modes propagating in the positive x-direction.

b. For a source frequency exciting 3 modes, make a sketch of the mode functions for pressure and for the particle velocity potential. Discuss the differences.

c. Derive the expression for the vertical wavelength of the modes.

d. Using the results from questions (a) and (c), state the lower limit for the vertical wavelength of a mode at angular frequency ω.

e. Use the result from (d) to determine how many modes you have at frequency $f = 30\,\mathrm{Hz}$.

2.11. In Eq. (2.170), $a_m(k_{rm})$ represents a waveguide-specific modal excitation function.

a. Derive the expression for $a_m(k_{rm})$ for the Pekeris waveguide.

b. Show that the modal excitation function has its maximum at the Airy phase frequency, i.e., the frequency where the mode has its minimum group velocity.

c. Compute and plot vs frequency the magnitude of the excitation function for the first 3 modes in the Pekeris waveguide in Fig. 2.25. Discuss the results.

2.12. A storm has created a 1 m thick surface layer with a uniform distribution of small air bubbles. The fraction of the volume occupied by the bubbles is 10^{-3}.

a. What assumption(s) do you have to make to treat the bubble layer as a homogeneous acoustic medium?

b. Under these assumptions, find the numerical values of the sound speed c and density ρ of the bubble layer. The sound speed of water and air are $c_w = 1500\,\mathrm{m/s}$ and $c_a = 340\,\mathrm{m/s}$, respectively, and the corresponding densities are $\rho_w = 1000\,\mathrm{kg/m^3}$ and $\rho_a = 1.2\,\mathrm{kg/m^3}$.

c. Show that the characteristic equation for normal modes in the bubble layer is

$$\cot(k_z h) = -\frac{\alpha_w}{k_z}\frac{\rho}{\rho_w}\ ,$$

where h is the thickness of the bubble layer, and

$$\begin{aligned} \alpha_w &= \sqrt{k_r^2 - \left(\frac{\omega}{c_w}\right)^2}\,, \\ k_z &= \sqrt{\left(\frac{\omega}{c}\right)^2 - k_r^2}\,. \end{aligned}$$

d. Discuss the physical significance of α_w and k_z.

e. What is the value of the cutoff frequency below which no normal modes can exist in the bubble layer?

2.13. In seismics, volume attenuation is often expressed in terms of the *quality factor*, defined as the ratio between the real and the imaginary part of the bulk modulus, i.e., $Q = K'/K''$ for $K = K' - iK''$. For small attenuations, ($Q \gg 1$), derive the relation between Q and the loss tangent δ, and the loss factor α in dB per wavelength.

2.14. Consider the reflection of plane waves from a bottom with the sound speed profile

$$c(z) = \begin{cases} (az+b)^{-1}, & 0 < z < 100\,\text{m}\,, \\ 1600\,\text{m/s}, & z \geq 100\,\text{m}\,. \end{cases}$$

The sound speed is continuous at the seabed ($z = 0$) and at $z = 100$ m, and the speed of sound in the water column ($z < 0$) is 1500 m/s.

a. Determine the constants a and b.

b. What is the critical grazing angle for waves incident from the water column?

c. Use the WKB approximation to derive expressions for the magnitude and phase of the reflection coefficient. Derive the result for grazing angles smaller and larger than critical. *Hint:*

$$\int \sqrt{\alpha + \beta x^2}\,dx = \frac{1}{2}\left[x\sqrt{\alpha+\beta x^2} + \frac{\alpha}{\sqrt{\beta}}\log\left(x\sqrt{\beta} + \sqrt{\alpha+\beta x^2}\right)\right].$$

d. For a frequency of 100 Hz, compute the phase of the reflection coefficient at grazing angles of incidence $30°, 40°, 50°, 60°, 70°, 80°$, and make a sketch of the result.

Chapter 3

Ray methods

3.1. Assume a deep ocean is represented by an infinite halfspace with a linear sound speed profile

$$c(z) = az + b, \quad a > 0.$$

A high frequency source is radiating from a point $(r, z) = (0, h)$. Consider a ray emitted from the source at grazing angle θ_0.

a. Derive a parameter representation for the ray path, before the first surface bounce,

$$\begin{aligned} r &= r(\theta, \theta_0) \ , \\ z &= z(\theta, \theta_0) \ , \end{aligned}$$

where θ is the local grazing angle for the ray.

b. Show that the ray path describes a circular arc and that the center of the circle falls at a depth $z = z_c$, where z_c is independent of the launch angle θ_0.

c. Derive the expression for the range $r_1(\theta_0)$, where the ray launched at angle θ_0 bounces off the sea surface.

d. Derive the parameter representation for the ray in the second ray cycle, i.e. after the first surface bounce.

e. Derive the expression for $dr/d\theta_0$ in the second ray cycle. Discuss the physical significance of the points where $dr/d\theta_0 = 0$.

3.2. An acoustic waveguide has the sound speed profile

$$c(z) = c_0 \cosh bz.$$

a. Show that for a source at $z = 0$, all rays will refocus at ranges $r = n\Delta r$ where n is an integer and Δr is independent of the launch angle. State the expression for Δr.

b. Discuss the physical significance of this phenomenon.

c. Write a simple ray code to demonstrate the refocusing.

d. Use your code to duplicate the result in Fig. 3.19.

3.3. Consider a source at depth $z_s = 2000\,\mathrm{m}$, range 0 and a receiver at depth $z_r = 4000\,\mathrm{m}$ and range $2\,\mathrm{km}$. Suppose that the sound speed depends only on depth, and that the values at the source and receiver depths are $1500\,\mathrm{m/s}$ and $1530\,\mathrm{m/s}$, respectively.

a. Use an n^2-linear approximation to estimate the travel time between source and receiver.

b. If the receiver is moved farther out in range there comes a point where the eigenray is turned before reaching the receiver. At what range does this first happen?

c. Is there a range where no real ray reaches the receiver? (Assume the water depth is infinite.)

3.4. Suppose we have a 500 Hz source launching a Gaussian beam in an isovelocity ocean with sound speed 1500 m/s.

a. If the beamwidth and curvature at the source are 100 m and zero respectively, what will the approximate beamwidth and curvature be 10 km away?

b. Suppose we want the beam to be as narrow as possible at 10 km. What initial beam width and curvature will do this?

c. Suppose the initial beam curvature has to be zero. What choice of the initial beam width will now give us the narrowest possible beam at 10 km?

3.5. A certain SSP has a sound speed of 1530 m/s at the surface, 1500 m/s at the source depth, 1550 m/s at the ocean bottom and 1800 m/s just below the bottom in the sediment. We will trace a fan of rays over angles $[-\theta, +\theta]$. How should we pick θ to include:

a. Only RR paths.

b. Only RR and RSR paths.

c. Only RR, RSR, and RSRBR paths striking the bottom with a grazing angle below the critical angle.

Chapter 4

Wavenumber Integration Techniques

4.1. Consider the reflection of a plane wave from an isovelocity fluid layer of thickness H overlying an isovelocity fluid halfspace for which $c_2 < c_1 < c_3$ and $\rho_1 < \rho_2 < \rho_3$.

a. What is the critical grazing angle for waves incident from medium 1?

b. If $k_2 H \ll 1$, show that to leading order the plane-wave reflection coefficient reduces to the plane-wave reflection coefficient without the layer present.

Now, suppose that $\rho_1 = \rho_3 < \rho_2$ and $c_1 = c_3 < c_2$, and that the plane wave is incident at grazing angle $\theta_1 < \arccos(c_1/c_2)$.

c. What is the angle of the transmitted wave in the lower halfspace, and what kind of wave is it (radiating or evanescent)?

d. What is the form of the solution in the layer?

e. Derive the expression for the reflection coefficient in the upper halfspace and the transmission coefficient in the lower halfspace.

f. By your intuition, what happens when $k_2 H \to \infty$? Verify your answer by examining the leading order behavior of the reflection and transmission coefficients.

4.2. Make a direct numerical implementation of the expression in Eq. (2.143) for the wavenumber representation of the field in an ideal waveguide. Allow the horizontal wavenumber to be complex.

a. For sound speed 1500 m/s and depth 100 m, compute the wavenumber kernel at 20 Hz for source and receiver both at depth 36 m. Sample the kernel at 200 points equidistantly placed over the interval $k_r \in [k_w/100, 2k_w]$, where k_w is the water wavenumber. Let the imaginary value of the horizontal wavenumber be $-k_w/100$ to avoid the modal singularities. NOTE: Your code will crash!

b. Determine the wavenumber interval for which your code produces a result which is qualitatively consistent with Fig. 2.23(a).

c. Describe the nature of the numerical problem, and rewrite Eq. (2.143) into a form which remedies the problem. Implement it and compare your result to Fig. 2.23(a) (qualitatively).

4.3. In matched field processing for source localization the sensitivity to environmental mismatch is a critical issue due to the fact that the environment is never known perfectly in a deterministic sense. The sensitivity to a sound speed perturbation in a stratified or range-independent ocean depends on the change in the depth-dependent Green's function associated with that perturbation. Let the wavenumber profile $k(z) = \omega/c(z)$ for such an ocean be given by a set of parameters $\mathbf{A} = [A_1, A_2, \ldots A_N]$. Show that the partial derivatives of the depth-dependent Green's function with respect to the parameter A_i are given by the depth integral

$$\frac{\partial G_\omega(k_r, z, z_s)}{\partial A_i} = \int_0^\infty \frac{\partial (k^2(z))}{\partial A_i} G_\omega(k_r, z_s, z')\, G^*_\omega(k_r, z, z')\, dz' \; .$$

4.4. The homogeneous displacement equation of motion in a homogeneous and isotropic elastic medium has the vector form,

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} = \rho \ddot{\mathbf{u}} \, .$$

a. Show that this equation is satisfied by displacement fields of the form,

$$\mathbf{u} = \nabla \phi + \nabla \times \Psi \, ,$$

where ϕ is a scalar potential satisfying the equation

$$\nabla^2 \phi - \frac{1}{c_P^2} \ddot{\phi} = 0 \, ,$$

and Ψ is a vector potential satisfying the equation

$$\nabla^2 \Psi - \frac{1}{c_S^2} \ddot{\Psi} = 0 \, ,$$

and where Ψ satisfies the *gauge condition* $\nabla \cdot \Psi = 0$.

b. Express c_P and c_S in terms of the Lamé constants λ and μ, and the density ρ.

c. What is the physical significance of the *gauge condition*?

d. Explain the physical significance of ϕ and Ψ.

4.5. Consider a homogeneous, isotropic and elastic halfspace with compressional speed c_P, shear speed c_S and density ρ.

a. For a plane compressional (P) wave incident on the free surface, derive the expressions for the reflected compression and shear potentials.

b. Discuss the existence of *total conversion* (no reflected P-wave) and *no conversion* (no reflected shear wave).

4.6. Consider a homogeneous, isotropic and elastic halfspace with compressional speed c_P, shear speed c_S and density ρ.

a. For a plane shear (SV) wave incident on the free surface, derive the expressions for the reflected compression and shear potentials.

b. Discuss the existence of *total conversion* (no reflected SV-wave) and *no conversion* (no reflected compressional wave).

4.7. Consider the problem of a water halfspace with sound speed c_1 and density ρ_1 overlying an elastic halfspace with compressional speed c_P, shear speed c_S, and density ρ_2.

a. Show that the depth-dependent Green's function for a point source in the water, at height H above the interface, has a denominator of the form,

$$d(k_r) = (2k_r^2 - k_S^2)^2 + 4k_r^2 k_{z,2}\kappa_{z,2} + k_S^4 \frac{\rho_1 k_{z,2}}{\rho_2 k_{z,1}} ,$$

where k_S is the shear wavenumber in the solid halfspace, k_r is the horizontal wavenumber and $k_{z,1}$ and $k_{z,2}$ are the vertical wavenumbers for compressional waves in the two media, and $\kappa_{z,2}$ is the vertical wavenumber for shear waves.

b. Show that $d(k)$ always has a real root k_{SCH},

$$k_{\mathrm{SCH}} > \max[k_1, k_S] ,$$

where k_1 is the wavenumber for acoustic waves in the water. The wave associated with this pole is called the *Scholte* wave.

c. Describe the frequency dispersion characteristics of the Scholte wave.

d. Make a sketch of the particle displacement associated with the Scholte wave on the surface of the elastic medium.

e. Assume the source is placed just above the bottom $H \simeq 0$, and emits a broadband signal. The field is measured by means of a bottom mounted vertical array far away from the source, where the field is dominated by the Scholte wave. If the frequency spectrum measured at the receiver on the interface is $F(\omega)$, what is the frequency spectrum at height h above the interface?

4.8. The denominator of the depth-dependent Green's function for the fluid–elastic halfspace problem described in the previous problem also has a symmetric pair of complex roots which become important for the propagation characteristics in certain cases.

a. Employ a numerical root finding scheme (e.g., a complex Newton–Raphson scheme) to determine the complex root with positive real value. (*Warning: take care how you choose the branch cuts for the square root*).

b. Assuming the sound speed in water to be 1500 m/s and a water density of 1000 kg/m^3, compressional speed 5000 m/s and density 2500 kg/m^3 in the solid, map the position of the root as function of shear speed in the range 1500–3500 m/s.

c. Discuss the physical significance of the real and imaginary part of the root.

4.9. An infinite elastic plate of thickness $2h$ is made of an elastic material with wave speeds c_P and c_S for compressional and shear waves, respectively, and density ρ_S. The plate is assumed to have free surfaces.

a. Show that the characteristic equation for the modes in the plate has the form

$$\frac{\tan(\kappa_z h)}{\tan(k_z h)} = -\left[\frac{4k_r^2 k_z \kappa_z}{(2k_r^2 - k_S^2)^2}\right]^{\pm 1} ,$$

where the "+" corresponds to symmetric modes and the "−" corresponds to antisymmetric modes. k_S is the shear wavenumber, and k_z and κ_z are the vertical wavenumbers for compression and shear, respectively. k_r is the horizontal wavenumber.

b. Show that in the low frequency limit,

$$\Omega = \frac{2h\omega}{\pi c_S} \rightarrow 0 ,$$

the characteristic equations reduce to,

$$\sinh \pi\gamma \ \pm \ \pi\gamma = 0 ,$$

where γ is a dimensionless horizontal wavenumber,

$$\gamma = \frac{2hk_r}{\pi} .$$

c. Solve the frequency equation numerically and graphically represent the $\omega - k_r$ relations for the first (fundamental) symmetric and antisymmetric modes for the elastic plate.

d. Discuss the cutoff properties and the static limits of the phase and group velocities for the two fundamental modes.

4.10. Equation (2.165) represents a DGM formulation for the Pekeris waveguide.

a. Is the direct numerical solution of Eq. (2.165) by Gaussian elimination numerically stable for all values of the horizontal wavenumber?

b. Modify Eq. (2.165) to make the solution unconditionally stable.

4.11. Consider an ocean waveguide similar the the Pekeris waveguide, but with an n^2-linear sound speed profile $c^2(z) = (az+b)^{-1}$ in the water column. Set up the corresponding global coefficient matrix in numerically stable forms for downward-refracting ($a > 0$) and upward-refracting ($a < 0$) profiles.

4.12. Assume you have to make a simple wavenumber integration code for propagation in Pekeris waveguides.

a. Make a subroutine which computes the wavenumber kernel, or depth-dependent Green's function, along a contour below the positive real wavenumber axis. Make sure your code is numerically stable for large wavenumbers, $k_r \gg (k_1, k_2)$.

b. Check your code by qualitatively reproducing the kernels shown in Fig. 4.7(a).

You decide to use FFP integration with a contour offset equal to the wavenumber sampling interval Δk_r.

c. What is the associated minimum attenuation of the wrap-around?

d. Using this offset, perform a numerical convergence analysis for the integration by computing the transmission loss at 46 m depth and 10 km range for the Pekeris waveguide in Fig. 2.25, assuming an attenuation of 0.5 dB/λ in the bottom. Note: You don't have to use FFT integration for this, use simple trapezoidal rule integration.

e. Repeat the convergence analysis without contour offset, and discuss the difference in convergence rate.

4.13. Develop an adaptive Filon integration scheme for general wavenumber integrals based on the FFP approximation (large argument Hankel function approximation).

a. Implement and test your algorithm using the Green's function subroutine developed for the previous problem.

b. Using the number of Green's function calculations as a performance measure, compare this approach to the simple direct trapezoidal rule integration in terms of computational efficiency for cases where the field is to be determined at a single range only.

c. Discuss qualitatively the performance of the adaptive scheme relative to use of an FFT to compute transmission loss at a large number of ranges.

Chapter 5

Normal Modes

5.1. Write a simple code to calculate the modes in a channel with a pressure-release surface and a rigid bottom. Compare your model results to those shown for the Munk profile in Fig. 5.10.

5.2. Consider a 300 m deep Pekeris waveguide with ocean sound speed of 1500 m/s and sediment sound speed of 1800 m/s.

   a. For a source frequency of 500 Hz, how many trapped modes are present? What are the horizontal wavenumbers for the first two modes?
   b. What is the cut-off frequency?
   c. What will the modes look like? (Sketch.)

5.3. For a certain frequency there is a mode for the Munk profile in Fig. 5.9 with phase speed 1535 m/s. Does it have an upper and lower turning point? If so, at what depth(s)?

5.4. Suppose we wish to write a normal mode code using Numerov's method.

   a. Write down a difference scheme to handle the ocean/sediment interface.
   b. What is the form of the final matrix of difference equations? (Assume a pressure-release surface and perfectly rigid bottom.)
   c. Discuss how you might solve the resulting algebraic eigenvalue problem.

5.5. How will the modes change across the eddy whose SSP is shown in Fig. 5.17 (Sketch).

5.6. Consider the following eigenproblem:

$$u'' + \lambda^2 u = 0\,,$$
$$u(0) + u'(0) = 0\,,$$
$$u(1) + u'(1) = 0\,.$$

The exact eigenvalues are $\lambda_k = k\pi$. If we solve this problem using finite differences with the standard formula, we will get approximations to these eigenvalues $\beta_k(N) = 2N \sin \frac{k\pi}{2N}$ where N is the number of points in the mesh and $k = 1, \ldots, N-1$.

a. How can we use our formula for $\beta_k(N)$ to obtain a similar result for the approximate eigenvalues of an isovelocity acoustic problem?

b. Calculate $\beta_1(10)$, $\beta_1(20)$, $\beta_1(40)$.

c. Use Richardson extrapolation to estimate $\beta_1(N)$ from these numbers in the limit $N \to \infty$.

d. Roughly, how large would N have to be to obtain this value by simple mesh refinement?

e. How much slower would the mesh refinement be? (Solving the finite difference equations for $\beta(N)$ requires roughly $20N$ operations.)

5.7. Ray-mode analogy: Consider a isovelocity waveguide bounded above and below by pressure-release surfaces.

a. Draw a diagram (see Fig. 2.2) with a "ray" reflecting with phase change, first from the bottom, and then from the surface. Construct a wavefront perpendicular to this ray such that it intersects both the ray when it is incident on the bottom and after it is reflected from the surface. What is the condition for angle and frequency that this wavefront be the result of perfect constructive interference?

b. What are the normal modes and eigenvalues of a waveguide with the above boundary conditions? (Note that Sec. 5.4 discusses the rigid bottom case).

c. Compare the two results.

d. Now assume that the bottom is a fluid and consider a ray more grazing than critical. It will be perfectly reflected but will undergo a phase change at the bottom given by the results in problem 1.5. What is the condition for perfect constructive interference. Compare this result with Eq. (5.80).

e. Which is a better approximation of a shallow water environment: a waveguide with a rigid or pressure-release bottom?

5.8. An alternative to using standard perturbation theory to compute the mode attenuation coefficients is to use a reflection coefficient argument. For an isovelocity waveguide, assume the magnitude of the bottom reflection coefficient to be close to unity, i.e., approximately $|R| = 1 - \epsilon$.

a. Derive an expression for the cycle distance associated with a mode. Using this cycle distance, express the change in the acoustic field as a function of the acoustic field itself, the cycle distance and the loss per bounce. This simple differential equation gives the modal attenuation coefficient.

b. What happens for the non-isovelocity case? Compute a skip distance by taking advantage of the fact that the horizontal wavenumber of a mode is constant whereas the vertical wavenumber varies with depth.

5.9. Another technique to compute bottom attenuation, which works for non-isovelocity cases is to assume a thin isovelocity layer just above the bottom. In this layer, normal modes are represented by up and down going waves with a reflection coefficient which includes the bottom attenuation term as in the problem above. The field and its derivative must be continuous in the water column. Take the limit of zero layer thickness to obtain the ratio of the normal mode to its derivative in terms of the reflection coefficient. Assume the modes and wavenumbers are complex and write down the eigenvalue equation and its complex conjugate. Multiply these equations by their complex conjugate mode function, respectively. Taking the difference of these two equations and integrating by parts will yield a relation connecting the imaginary part of the wavenumber with the normal mode and its derivative. Use this method to derive an expression for the modal attenuation coefficient.

5.10. The technique of the last problem can be used to approximate the effects of a low-shear-speed bottom. In this case, a shear wave is an additional mechanism to transmit sound out of the water column; hence, it acts as a loss mechanism.

a. Use a small parameter expansion of the fluid–elastic reflection coefficient to derive the effective modal attenuation coefficient due to the existence of a low shear speed c_S in the bottom sediment.

b. At what shear speed do you expect this approximation to break down?

5.11. Ocean currents affect sound propagation. For simplicity consider sound from a line source propagating in a laminar flow velocity $V(z)$ parallel to the ocean bottom and in the positive x-direction. Linearizing about the background state as in Sec. 2.1 one can derive the following convected wave equation:

$$\begin{aligned}\rho\,(u_t + V u_x + w V_z) &= -p_x\,,\\ \rho\,(w_t + V w_x) &= -p_z\,,\end{aligned}$$

$$p_t + V p_x + c^2 \rho \left(u_x + w_z \right) \;\; = \;\; 0 \, ,$$

where u and w are the acoustic particle velocities in the x and z-directions, respectively.

a. Show that the normal modes of this equation satisfy

$$\left[\frac{1}{\left(\omega - kV\right)^2} \psi_z \right]_z + \left[\frac{1}{c^2} - \frac{k^2}{\left(\omega - kV\right)^2} \right] \psi = 0 \, .$$

Note that $V(z) = 0$ gives the usual modal equation.

b. Ocean currents will satisfy a no-slip condition implying that the flow velocity vanishes at the bottom. Nevertheless, consider an ocean with uniform flow, uniform sound speed and with a perfectly rigid bottom. What is the dispersion relation? Plot representative curves for different modes and flow speeds. Include the asymptotes.

Chapter 6

Parabolic Equations

6.1. The standard parabolic wave equation can be derived by introducing a narrow-angle approximation to a modal representation of the field in a waveguide. Let the modal solution be given by

$$p(r,z) = \sum_m a_m \Psi_m(z) \frac{e^{ik_m r}}{\sqrt{k_m r}} ,$$

where the eigenfunctions $\Psi_m(z)$ satisfy the depth-separated wave equation

$$\frac{d^2\Psi_m(z)}{dz^2} + \left[k_0^2\, n^2(z) - k_m^2\right] \Psi_m(z) = 0 .$$

Here k_0 is the reference wavenumber and $n(z) = k/k_0$ the index of refraction. By assuming the modal eigenvalues to cluster around k_0 (a narrow-angle approximation) and to be given in the form $k_m = k_0(1-\epsilon_m)^{1/2}$, where ϵ_m is small compared to unity, show that to leading order in ϵ_m the field solution can be written in the form $p(r,z) = \psi(r,z)\exp(ik_0 r)/(k_0 r)^{1/2}$, where the envelope function $\psi(r,z)$ satisfies the standard parabolic equation (6.8).

6.2. Derive a three-dimensional parabolic wave equation in cylindrical coordinates (r, φ, z) and show that it reduces to Eq. (6.8) for no azimuthal dependence of the refraction index n.

6.3. The effect of earth curvature on long-range propagation in the ocean can be easily accounted for in acoustic models via a modification of the local sound-speed profile.

   a. With r being the horizontal range from a source and R the earth radius, show that the sea surface on a sphere is displaced by $\Delta z \simeq r^2/2R$.

   b. By introducing the transformation

$$\psi(r,z) = \psi'(r,z')\ \exp\left[ik_0 r\left(\frac{z'}{R} - \frac{r^2}{6R^2}\right)\right] , \qquad z' = z - \Delta z ,$$

   and substituting into Eq. (6.8), derive a parabolic wave equation in $\psi'(r,z')$.

   c. Discuss the form of this equation and show that the earth curvature effect can be accounted for by a small linear increase in sound speed with depth.

   d. Estimate the percentage change in convergence-zone ranges due to earth curvature.

6.4. Rayleigh's principle for one-way wave propagation asserts that the average kinetic energy in the wave must be equal to the average potential energy, i.e.,

$$\int \frac{1}{4}\left(|u|^2 + |v|^2\right)\, dz = \int \frac{1}{4}\rho c^{-2} |p|^2 \, dz \, .$$

Here u is the horizontal particle velocity, v the vertical particle velocity, and p the pressure. This energy conservation formula can be used to determine a "natural" reference wavenumber k_0 for propagation in any of the parabolic approximations to the Helmholtz equation.

a. Derive an approximate expression for k_0 in terms of integrals of field quantities satisfying the standard parabolic equation (6.26).

b. For a single mode propagating in an ideal, pressure-release waveguide show that the "natural" wavenumber found in (a) equals the modal eigenvalue.

c. Discuss the implications of multi-mode propagation for the choice of a reference wavenumber, particularly in lossy environments with mode stripping.

d. Consider next the alternative PE form given by Eq. (6.40). Derive the approximate expression for k_0 and show that for single-mode propagation in an ideal, pressure-release waveguide the "natural" wavenumber now equals the water wavenumber.

6.5. Consider upslope propagation in an isovelocity wedge as illustrated in Fig. 6.11.

a. Under the assumption of no bottom attenuation, derive an expression for the number of modes present in such a Pekeris waveguide. *Hint:* Use the information given in Sec. 2.4.5.

b. Calculate the nominal cutoff ranges (depths) for the three propagating modes in Fig. 6.11 and compare with the PE-generated field solution.

Chapter 7

Finite Differences and Finite Elements

7.1. For a two-dimensional finite difference grid of spacing h in x and y, derive an $O(h^2)$ finite difference approximation to the derivative

$$\frac{\partial^2 u}{\partial x \partial y} .$$

7.2. Consider the one-dimensional wave equation

$$\frac{\partial^2 p(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial t^2} = 0 ,$$

subject to the boundary conditions

$$\begin{aligned} p(0,t) &= P(t) , \\ \left.\frac{\partial p(x,t)}{\partial x}\right|_{x=D} - \alpha p(D,t) &= 0 . \end{aligned}$$

You may assume the sound speed and density is constant for $x \in [0, D]$.

a. Show that the boundary condition at $x = D$ represents the reflection from a plane interface separating two fluid media.

b. Assume $[0, D]$ represents an acoustic medium with $c = 1500\,\mathrm{m/s}$ and $\rho = 1000\,\mathrm{kg/m^3}$, and that D represents an interface to an acoustic halfspace with $c_2 = 1600\,\mathrm{m/s}$ and $\rho_2 = 1800\,\mathrm{kg/m^3}$. Find the corresponding value of α.

c. Assume the boundary pressure $P(t)$ is of the form

$$P(t) = \begin{cases} 0, & t \le 0 \\ 1 - \cos^2(4\pi t c/D), & t < D/(4c) \\ 0, & t \ge D/(4c) . \end{cases}$$

Derive the analytical solution for $p(x,t)$ for $t \in [0, 2D/c]$.

7.3. Make a finite difference code for solving the previous problem for $D = 1500\,\mathrm{m}$. Choose a simple explicit scheme similar to that described in Sec. 7.3.4.

a. Perform a numerical convergence analysis and compare your results to the analytical result.

b. Show that the convergence rate is consistent with the order of the finite difference approximations used.

7.4. Assume you want to solve the one-dimensional Helmholtz equation (7.72), with homogeneous boundary conditions $p(0) = p(D) = 0$, using FEM with global trial functions.

a. Determine the set of trial functions which yield a diagonal coefficient matrix for a homogeneous medium.

b. Determine the FEM solution for a point source at $x = x_s$ in the case of a homogeneous medium.

c. Discuss the relation between this solution and the normal mode solution for an ideal waveguide, Sec. 2.4.4.

7.5. Consider a fluid waveguide similar to the Pekeris waveguide, but with a continuously varying sound speed $c(z)$ in the water column.

a. Using the simple linear elements shown in Fig. 7.7, formulate the FEM equations for the depth-separated wave equation. You may assume the sound speed to be linear (but not constant) within each element.

b. Discuss the factors affecting the choice of element size for this problem.

c. Implement the formulation and perform a numerical convergence analysis for the isovelocity Pekeris waveguide. Compare your results to the analytical solution [e.g., by solving Eq. (2.165)].

7.6. Assume a finite element mesh is composed of triangular elements which are all identical, but rotated versions of the one shown in Fig. 7.9. When setting up the global finite element equations, all node displacements must be aligned with the coordinate axes as shown in Fig. 7.9.

a. Assume the stiffness matrix $\mathbf{k}$ for one of the elements has been determined. Show that the stiffness matrix for another, rotated element can be determined by an expression of the form,

$$\mathbf{k}^* = \mathbf{A}^T \mathbf{k} \mathbf{A} \,,$$

b. Derive the expressions for the coefficients of $\mathbf{A}$ for an element rotated by an angle θ.

7.7. Assume you have to write a finite element code solving the Helmholtz equation in a rectangular domain using the following mesh of simple, triangular elements with nodes in the corners:

a. Describe the strategy you would use for setting up the local element matrices.

b. Select a local element numbering for your elements, and determine the global node numbering which yields the minimum bandwidth of the global coefficient matrices.

c. Write out the topology matrix **L** corresponding to the numbering you selected.

Chapter 8

Broadband Modeling

8.1. Defining the *bandwidth* of a source wavelet as the total width of the main lobe of its frequency spectrum, show that the bandwidth of the wavelet

$$S(t) = \begin{cases} \frac{1}{2}\sin 2\pi f_c t\,(1-\cos\frac{\pi}{2N}f_c t) & \text{for } 0 < t < 4N/f_c \\ 0 & \text{else} \end{cases},$$

is equal to f_c/N.

8.2. Using Fourier synthesis you have to compute a field produced by the source wavelet

$$S(t) = \begin{cases} \sin(2\pi f_c t) - \frac{1}{2}\sin(4\pi f_c t) & \text{for } 0 < t < 1/f_c \\ 0 & \text{else} \end{cases}.$$

a. Determine the frequency spectrum $S(f)$ of this wavelet.

b. At which frequency $f_{\max}$ would you truncate the computation of the Green's functions? Justify your answer.

c. The maximum time duration of the impulse response is $T_I = 15/f_c$. What is the frequency sampling required to avoid wrap-around in the computed response?

8.3. Assume you have a code for computing the transfer function $p(r, z, \omega)$ for the reflection problem in Fig. 8.2, which you want to use together with Fourier synthesis to model the transient response on a horizontal receiver array 100 m above the interface. The array has 11 elements at a spacing of 50 m, with the first element at $r = 0$.

a. Which array elements will record the *head wave*?

b. If you use a *fixed* time window, starting at time $t = 0$, determine the minimum length T_f of the time window necessary to avoid wrap-around of the response of any of the receivers.

c. Similarly, determine the minimum length T_r of the time window if you allow the starting time $t_{\min}$ to be receiver-dependent (*running* time window).

d. In general, the computation time for the Fourier synthesis is insignificant compared to that associated with the computation of the transfer functions. Determine in relative terms the computational advantage of using the *running* time window for this problem.

e. Would there be any computational advantage in using a receiver-dependent length of the window as well?

8.4. Write a computer program for solving the reflection problem described in Problem 1 above. You may use library routines where feasible.

a. Discuss your selection of time and frequency sampling.

b. Make a plot of the stacked time series using a *running* time window with $t_{\min} = r/2500$.

c. Identify the various arrivals on the plot, and discuss any possible differences in pulse shape.

8.5. A source and a receiver are moving horizontally in a horizontally stratified ocean with velocity vectors $\mathbf{v}_s$ and $\mathbf{v}_r$, respectively.

a. Show that in the frequency domain, the field observed by the receiver is given by the expression

$$\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) = \frac{1}{2\pi} \int d^2\mathbf{k}_r e^{i\mathbf{k}_r \cdot \mathbf{r}_0} S(\Omega_k)\, G(k_r, z; \omega + \mathbf{k}_r \cdot \mathbf{v}_r)\,,$$

where Ω_k is the Doppler shifted source frequency

$$\Omega_k = \omega - \mathbf{k}_r \cdot (\mathbf{v}_s - \mathbf{v}_r)\,.$$

b. Discuss the computational advantages of using this representation together with Fourier synthesis to determine the time domain solution, rather than using Eq. (8.58) directly.

c. Derive the modal representation for the frequency domain solution.

8.6. It is desired to send out an n-cycle CW pulse of center frequency f_c in shallow water such that the modes are temporally separated at range r. Using group velocity arguments, determine the relationship between f_c, n (taken together, bandwidth) and r for the onset of this mode separation. Confirm this with a numerical computation.

Chapter 9

Ambient Noise

9.1. Let the ocean be a semi-infinite isovelocity halfspace bounded above by a uniform distribution of monopole sources radiating with an intensity per unit area at a unit distance. What is the depth dependence of the intensity of the noise field? Now assume that the spreading law is cylindrical rather than spherical. What additional physical parameter must be included to give physically sensible results?

9.2. Define directionality of the noise field to be the noise intensity per unit solid angle. Derive an expression for the noise directionality in the ocean described in the above problem. How does the result change if the sources are dipoles rather than monopoles? (Take the intensity radiation pattern for a dipole to be proportional to $\cos^2\theta$ where θ is the angle measured from the normal to the surface).

9.3. The cross-spectral density and the directionality are related by a Fourier transform. What are the Fourier conjugate variables? Compare the monopole and dipole results derived in the last problem with the Cron and Sherman results discussed in Sec. 9.2.5.

9.4. Consider a sonar receiver array with baffled sensors which individually have a beam pattern $W(\theta,\phi) = W_1(\theta)\,W_2(\phi)$, where θ is the vertical angle and ϕ is the azimuthal angle. The sonar is used in a stratified ocean with a uniform distribution of surface noise sources.

   a. Derive an expression for $C_\omega(\mathbf{r}_1, \mathbf{r}_2, z_1, z_2)$, the cross-spectral density function for the ambient noise as seen by the array.

   b. Show that your result is consistent with the result of Kuperman and Ingenito for $W(\theta,\phi) \equiv 1$.

9.5. Develop an algorithm for generating a realization of noise time series for a receiver array in a stratified ocean with ambient noise generated by a homogeneous distribution of surface noise sources.

Chapter 10

Signals in Noise

10.1. Assume you are using a long horizontal array for passively detecting a sound source in the ocean. The array characteristics are as follows:

Length:	4050 m
Element spacing:	75 m
Number of elements:	55

To estimate the source bearing in deep water it is often a good approximation to perform the beamforming assuming the source and the array are at equal depth in an infinite, homogeneous medium with the sound speed equal to the one existing at the array depth (assume 1500 m/s).

a. Under such conditions write a linear beamformer algorithm for estimating source bearing, and use it to compute the beamformer response to a 10-Hz point source at bearing 45° off broadside, at a range of 80 km from the center of the array.

b. Discuss the features of the beamformer response.

c. Compute the corresponding beamformer response at 30 Hz, and discuss the result.

10.2. Assume you have to use the array from the previous problem in an isovelocity (1500 m/s), shallow water environment with water depth 120 m, and with an infinitely rigid bottom. Assume you are towing the array at 60-m depth, and that the source is at 60-m depth as well.

a. For a 10-Hz source at 45° bearing, and 80-km range, write an algorithm for computing the field on the elements of the array in terms of a modal expansion.

b. Use the linear beamformer developed for Problem 1 to compute the response.

c. Discuss the features of the beamformer response and provide a physical explanation for the performance.

d. How do you suggest to modify the beamformer to yield a correct bearing estimate?

e. Implement the modification and discuss the performance.

10.3. From problem 9.3 we learned that directionality is related to the Fourier transform of the cross-spectral density. In this chapter, we note that plane wave beamforming is a finite spatial Fourier transform of the acoustic field with sensor spacing being one of the conjugate Fourier variable. Using a vertical array, beamform on a range-independent shallow water noise field with sufficient resolution to show that there is a "horizontal notch." Physically, why must such a notch exist?

10.4. For an acoustic field in a waveguide, how would you take advantage of the orthogonality of normal modes to construct a modal, rather than plane wave beamformer? Use a vertical array.

10.5. Write down an expression in terms of discrete normal modes for the field of a point source received on a vertical array. Take the complex conjugate of this result and use the result to represent a distribution of point sources. With this source distribution and the known Green's function of this problem, propagate the resulting field outward. (With this range independent geometry, outward is the same as "backward.")

a. What happens at the position of the original point source?

b. How does the vertical array geometry affect the results?

c. This method is called backpropagation. Is there a difference between this method and Bartlett matched field processing?