A Legendre-Galerkin technique for differential eigenvalue problems with complex and discontinuous coefficients, arising in underwater acoustics

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ABSTRACT:

The depth separated wave equation in underwater acoustics is a non-selfadjoint differential eigenvalue problem with complex and discontinuous coefficients. A Galerkin approach using domain decomposition and a spectral method with a basis of Legendre polynomials leads to a complex symmetric generalized matrix eigenvalue problem. The eigenvector solutions are found to satisfy a weighted biorthogonality condition that assures the proper normalization of the corresponding acoustic modes and the acoustic field.

I. INTRODUCTION:

The depth separated wave equation for the normal modes in underwater acoustics[1] can be solved by both Fourier-Galerkin and Legrendre-Galerkin methods[2]. This terminology requires a more generic explanation. The Galerkin method has several main features: Its application involves the projection of the problem onto a subspace spanned by a finite dimensional basis. The projection is applied to a partially integrated form of the differential equation. The basis elements are required to satisfy the boundary and interface conditions associated with the differential equation and the projection yields a problem in linear algebra. In contrast to finite elements, the basis elements are global rather than localized in space. The linear algebra problem is usually smaller with global elements.

The basis elements are constructed using the spectrum of a differential operator with readily computable spectral elements. When these elements are trigonometric functions, the term Fourier-Galerkin is used. When these elements are orthogonal polynomials, the Galerkin method inherits the name of the class of orthogonal polynomials. This is the case with Legendre polynomials. Both Legendre-Galerkin and Fourier-Galerkin are special cases of spectral methods pioneered by Gottlieb and Orsag[3]. A historical background for spectral methods is found in Canuto et al.[4]. The Legendre-Galerkin method is of particular interest since the underlying differential operator is singular, rendering an enhanced rate of convergence for the spectral expansion[3].

The trigonometric functions and orthogonal polynomials will not automatically satisfy the boundary and interface conditions, so a form of basis recombination and domain decomposition must be used. Enforcing the boundary and interface conditions in the Fourier-Galerkin method requires the solution of a transcendental equation while this can be accomplished algebraically in the Legendre-Galerkin method. See Section IV.

II. STATEMENT OF PROBLEM:

The depth separated wave equation, for the harmonic acoustic pressure, has the form of a differential eigenvalue problem given by[1]

$$\frac{d}{dz}\left[\frac{1}{\rho(z)}\frac{d\phi}{dz}\right]+\frac{k^2(z)}{\rho(z)}\phi=\frac{\kappa^2\phi}{\rho(z)} \tag{1}$$

where κ^2 is a separation constant. The separation constant is the square of the horizontal wave number and its values are determined by the coefficients in the differential equation, the boundary conditions at the ends of the interval $0 \leq z \leq H$ and the interface conditions at $z = h$ where $h < H$. The geometry of the problem is shown in Fig. 1. The function ρ is the density and $k = (\omega/c)[1+i(\alpha/40\pi \log_{10} e)]$ is the complex wave number, where ω is the circular frequency, c is the sound speed and α is the attenuation in dB per wavelength. The functions ρ and k^2 can be discontinuous at the bottom of the water column, at the interface where $z = h$.
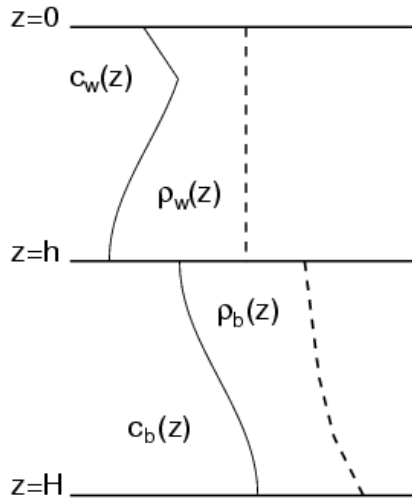

Figure 1. Sound speed and density profiles for the depth separated wave equation.

The complex wave number squared and density, in Eq. (1), are defined in the water and bottom, separately, using the non-negative real functions

$$c(z)=\begin{cases} c_w(z),\ 0\leq z\leq h \\ c_b(z),\ h<z\leq H \end{cases},\ \alpha(z)=\begin{cases} \alpha_w(z),\ 0\leq z\leq h \\ \alpha_b(z),\ h<z\leq H \end{cases},\ \rho(z)=\begin{cases} \rho_w(z),\ 0\leq z\leq h \\ \rho_b(z),\ h<z\leq H \end{cases}.$$

An example of a sound speed and density profile is shown in Fig. 1.

The boundary conditions are $\phi(0) = 0$ and $\phi(H) = 0$, which correspond to a free (pressure release) surface at $z = 0$ and H. The continuity of pressure and particle velocity dictate that $\phi(h^-) = \phi(h^+)$ and $[1/\rho(h^-)]d\phi(h^-)/dz = 1/\rho(h^+)]d\phi(h^+)/dz$ where the superscripts – and + indicate limits from above and below, respectively.

The differential eigenvalue problem consisting of Eq. (1) and associated boundary and interface conditions is non-selfadjoint because the wave number squared is complex. The eigenvalues and eigenfunctions are generally complex. It is assumed that the attenuation is small enough so that the eigenvalues are simple roots of the characteristic equation. The problem is an analytic perturbation[5] of the real selfadjoint case, without any attenuation.

A Sturm-Liouville form of the problem in Eq. (1) is obtained by translating the separation constant and changing its sign as follows: Subtract $k_{max}^2 \phi / \rho(z)$ from both sides of Eq. (1) where $k_{max} = \omega / \min\{c(z) : 0 \le z \le H\}$. Multiply by negative to obtain

$$-\frac{d}{dz}\left[\frac{1}{\rho(z)}\frac{d\phi}{dz}\right] + \frac{q(z)}{\rho(z)}\phi = \frac{\lambda\phi}{\rho(z)} \qquad (2)$$

where $q(z) = k_{max}^2 - k^2(z)$ and $\lambda = k_{max}^2 - \kappa^2$. The motivation for this procedure is to effect a sign change on the terms in Eq. (3) below that is especially useful in the real selfadjoint case. In the non-selfadjoint case, the solutions to Eq. (2) consist of a countably infinite sequence of complex numbers λ_m called eigenvalues and complex functions ϕ_m called eigenfunctions (or modes) that satisfy Eq. (2) and the stated boundary and interface conditions. In an analogy with the real selfadjoint case, it will be assumed that the complex eigenvalues λ_m for , $m = 1, \infty$ are ordered by increasing real parts. The horizontal wave numbers squared are $\kappa_m^2 = k_{max}^2 - \lambda_m$, $m = 1, \infty$.

III. GALERKIN METHOD:

A comprehensive overview of the Galerkin method, starting from the method of mean weighted residuals, can be found in Boyd[6]. The Galerkin method is an approximation to a partially integrated form of Eq. (2). To this end, suppose that χ is a continuous function that is piecewise differentiable on intervals $0 \le z \le h$ and $h \le z \le H$ and satisfies the same boundary and interface conditions as the eigenfunction ϕ. The name "test function" is used for the class of functions with these properties. Multiplying Eq. (2) by χ and integrating by parts over the interval $0 \le z \le H$ yields

$$\int_0^H \frac{q(z)\chi(z)\phi(z)}{\rho(z)}dz + \int_0^H \frac{1}{\rho(z)}\frac{d\chi(z)}{dz}\frac{d\phi(z)}{dz}dz = \lambda\int_0^H \frac{\chi(z)\phi(z)}{\rho(z)}dz \ . \qquad (3)$$

The boundary and interface conditions eliminate the boundary and interface terms in the derivation of Eq. (3). Equation (3) is referred to as the weak form of the Sturm-Liouville problem in Eq. (2). Equation (3) will be abbreviated as $a(\phi, \chi) = \lambda b(\phi, \chi)$ where the bilinear forms $a(\phi, \chi)$ and $b(\phi, \chi)$ are the integrals on the left and right hand sides of Eq (3), respectively.

In the selfadjoint case, when $q(z) > 0$, using the real eigenfunctions as test functions in the integral relation $a(\phi, \phi) = \lambda b(\phi, \phi)$ shows that the eigenvalues are positive real numbers. The eigenvalues increase in size, in proportion to norm of the derivative of the eigenfunctions.

The Galerkin method is predicated on a basis[7], which is a linearly independent set of functions that span a finite dimensional space. When the space is the set of all linear combinations of the functions, it is sufficient that they be linearly independent. One way to assure linear independence is to use orthogonality, but orthogonality is not necessary. Requiring only linear independence simplifies the choice of a spanning set, but can result in very poorly conditioned matrices[8]. Boyd[6] calls orthogonality a maximization of linear independence. The distinction between Galerkin methods based on orthogonal and non-orthogonal or oblique bases is made in Chatelin[9].

In the current application of the Galerkin method, an eigenfunction of Eq. (2) is approximated by the expansion $\phi \cong \sum_{n=1}^{M} u_n \psi_n$ where $\{\psi_n : n = 1, M\}$ is a linearly independent set of test functions. It is assumed that these test functions are real while the expansion coefficients may be complex. Substituting the expansion into Eq. (3), with $\chi = \psi_m$, yields the discrete eigenvalue problem $\sum_{n=1}^{M} a_{m,n} u_n = \mu \sum_{n=1}^{M} b_{m,n} u_n$ where $a_{m,n} = a(\psi_m, \psi_n)$ and $b_{n,m} = b(\psi_n, \psi_m)$ and μ is a discrete eigenvalue. Note that $a_{m,n}$ is complex when the wave number squared is complex and $b_{n,m}$ is real since the density is real. The Galerkin method turns Eq. (3) into a discrete problem by restricting it to the subspace spanned by the basis $\{\psi_m : m = 1, M\}$.

The bases, used in the Galerkin method, determine the accuracy of the approximations to the eigenvalues and eigenfunctions of Eq. (2). In particular, it is important that the basis functions have the same derivative discontinuity as the eigenfunctions in order to avoid Gibbs oscillations[3]. A systematic theoretical study of the convergence of the Galerkin method is found in Babuška and Osburn[10]. Examples of the asymptotic rate of convergence in the Fourier-Galerkin and Legendre-Galerkin approximations have been previously reported[2]. Since the orthogonal Fourier-Galerkin method has been described elsewhere[2], the non-orthogonal Legendre-Galerkin method will be described in detail below. Both methods share the same features in their matrix formulation. These characteristics are presented first. The integer M determines size of the discrete problem and accuracy of the approximations

A. Matrix Eigenvalue Problem:

Let $\mathbf{A}$ and $\mathbf{B}$ be MxM matrices whose entries are $a_{m,n}$ and $b_{m,n}$, respectively. If a column vector is defined by $\mathbf{u} = col(u_n, n = 1, M)$, then the discrete problem can be written as the generalized matrix eigenvalue problem

$$\mathbf{Au} = \mu\,\mathbf{Bu} \qquad\qquad (4)$$

where $\mathbf{A}$ is a complex symmetric (not Hermitian symmetric) matrix and $\mathbf{B}$ is a real symmetric matrix. The eigenvalues of Eq. (4) approximate the eigenvalues of Eq. (2). The eigenvectors of Eq. (4) provide the expansion coefficients for approximates of the eigenfunctions of Eq. (2).

Equation (4) can be turned into a standard matrix eigenvalue problem by multiplying on the left by $\mathbf{B}^{-1}$, but this destroys the symmetry of the problem. If $\mathbf{B}$ is positive definite, then Eq. (4) can be replaced by the complex symmetric eigenvalue problem

$$\mathbf{Cw} = \mu\,\mathbf{w} \qquad\qquad (5)$$

where $\mathbf{C} = \mathbf{L}^{-1}\mathbf{A}(\mathbf{L}^T)^{-1}$, $\mathbf{w} = \mathbf{L}^T\mathbf{u}$, and $\mathbf{B} = \mathbf{LL}^T$ is the Cholesky decomposition of $\mathbf{B}$. The superscript T stands for transpose (without complex conjugation) and $\mathbf{L}$ is a lower triangular matrix, making the calculation of $\mathbf{C}$ relatively easy[11]. Note that $\mathbf{u}^T$ is a row vector and its product with the column vector $\mathbf{v} = col(v_j, j = 1, M)$ is the scalar $\mathbf{u}^T\mathbf{v} = \sum_{j=1}^{M} u_j v_j$. Again, there is no complex conjugation in the transpose of the vector.

To see that $\mathbf{B}$ is positive definite, let $\mathbf{x} = col(x_j, j = 1, M)$ be a real vector and define the real function $f = \sum_{j=1}^{M} x_j \psi_j$. Since the basis functions are linearly independent, the function f cannot be identically zero, unless $\mathbf{x} = \mathbf{0}$. It also follows that $\int_0^H [f^2(z)/\rho(z)]dz = \mathbf{x}^T\mathbf{Bx}$. Consequently, $\mathbf{x}^T\mathbf{Bx} > 0$, unless $\mathbf{x} = \mathbf{0}$.

If the complex symmetric matrix $\mathbf{C}$ in Eq. (5) is diagonalizable[12], then $\mathbf{C} = \mathbf{W}\,\mathbf{\Lambda}\,\mathbf{W}^T$ where $\mathbf{\Lambda} = diag(\mu_m, m = 1, M)$ and $\mathbf{W}$ is an orthogonal matrix (not unitary) such that $\mathbf{W}^T\mathbf{W} = \mathbf{I}$ is the MxM identity matrix. The columns of $\mathbf{W} = row(\mathbf{w}_m, m = 1, M)$ are the eigenvectors of $\mathbf{C}$. The orthogonality of $\mathbf{W}$ implies that $\mathbf{w}_n^T\mathbf{w}_m = \delta_{n,m}$. The eigenvectors of Eq. (4) are recovered by solving $\mathbf{L}^T\mathbf{u}_m = \mathbf{w}_m$ and consequently $\mathbf{u}_n^T\mathbf{Bu}_m = \delta_{n,m}$, when $\mathbf{W}$ is orthogonal. For the sake of comparison, it is assumed that the approximate eigenvalues $\{\mu_m, m = 1, M\}$ are ordered by increasing real parts.

The matrix eigenvalue problem in Eq. (5) can be tackled by several methods. The Jacobi technique[13], programmed for complex symmetric matrices by Anderson and Loizou[14], has the advantage that the resulting matrix $\mathbf{W}$ is a product of rotations and is, consequently, orthogonal. The Jacobi technique is not the most efficient, but it is simple and convergent, assuming that $\mathbf{C}$ is diagonalizable.

The generalized matrix eigenvalue problem in Eq. (4) can also be solved by using the Matlab procedure $[\mathbf{V}, \mathbf{D}] = eig(\mathbf{A}, \mathbf{B})$. A matrix formulation of the eigenvalue problem associated with Eq. (1) can be obtained using differentiation matrices based on Chebyshev polynamials[15]. The Matlab approach, using a spectral collocation scheme, has been employed to computed eigenvalues and the corresponding dispersion relationships for layered elastic media[15,16]. Field calculations require the construction and appropriate normalization of the eigenfuntions.

B. Biorthogonality:

The construction of the fundamental solution of the two dimensional Helmholtz equation, by separation of variables, requires that the eigenfunctions be properly orthonormal[1]. When the complex eigenfunctions are used, the orthogonality condition is replaced by a biorthogonality condition. The weighted biorthogonality property of the complex eigenfunctions, is given by

$$\int_0^H \frac{\phi_n(z)\phi_m(z)}{\rho(z)}\,dz = \delta_{n,m} \ . \tag{6}$$

Although Eq. (6) is indistinguishable from the real orthogonality condition, the term biorthogonality is used because of the non-selfadjoint nature[17] of Eq. (1). In short, it means that the complex conjugate that would normally appear on one of the complex functions in the integrand is omitted. This conforms to the definition of the scalar product of two complex vectors, already introduced. The biorthogonality of the Galerkin approximates of the complex eigenfunctions is demonstrated as follows.

Consider two approximate eigenfunctions of Eq. (2) defined by $\varphi_m = \sum_{j=1}^{M} u_{m,j}\psi_j$ and $\varphi_n = \sum_{j=1}^{M} u_{n,j}\psi_j$, where $\mathbf{u}_m$ and $\mathbf{u}_n$ are two eigenvectors of Eq. (4). The integral in Eq. (6) can be evaluated by substitution of these two expansions to obtain

$$\int_0^H \frac{\varphi_n(z)\varphi_m(z)}{\rho(z)}\,dz = \mathbf{u}_n^T \mathbf{B}\mathbf{u}_m \ .$$

The biorthogonality of the approximate eigenfunctions follows from $\mathbf{u}_n^T \mathbf{B}\mathbf{u}_m = \delta_{n,m}$, since $\mathbf{W}$ is assumed to be orthogonal.

C. Acoustic Field:

   The approximations of the eigenvalues of Eq. (2) determine approximations of the separation constants in Eq. (1), and ultimately the normal modes appearing in the acoustic field. The calculation of the acoustic field relies on approximating the horizontal wave numbers κ_m by $\nu_m = \sqrt{k_{max}^2 - \mu_m}$, $m = 1, M$. The square root is taken to have a positive imaginary part to assure the proper attenuation with range. The fundamental solution of the 2d Helmholtz equation is approximated[1] by

$$p(r,z) \cong \left\{ \frac{i \exp(-i\pi/4)}{\rho(z_s)\sqrt{8\pi}} \right\} \sum_{m=1}^{MS} \varphi_m(z_s) \varphi_m(z) \frac{\exp(i\nu_m r)}{\sqrt{\nu_m r}} \tag{7}$$

where $MS \leq M$ is the number of modes to be used in the approximation of the acoustic field. The vector (r, z) contains the receiver range and depth, in meters, and z_s is the source depth.

   In the selfadjoint case the approximate eigenvalues μ_m are real. For smaller values of m, μ_m should be less than k_{max} and the horizontal wave numbers ν_m will be real. Eventually, μ_m should be greater than k_{max} and ν_m will be purely imaginary. These two distinct groups of horizontal wave numbers are called propagating and evanescent. The evanescent modes are unnecessary at large ranges. By analogy, in the non-selfadjoint case, the distinction is not as sharp. The imaginary parts of the horizontal wave numbers increase more gradually with m. The modes are be called water born and bottom interacting, since most of the attenuation is associated with the sediment. The integer MS should be chosen with consideration of what range interval is of interest.

   Clearly, the choice of M and MS requires an iterative process. Initially, confidence may be developed by comparison with other theoretical methods based on wave number integration or the parabolic equation.

IV. LEGENDRE POLYNOMIAL BASIS:

   A basis is a spanning set of linear independent functions. Linear independence assures that the projection onto the finite dimensional subspace is well defined (unique). The Legendre polynomial spanning set to be constructed is not manifestly independent, so the term basis is held in reserve. The construction of the real Legendre polynomial spanning starts with a choice of the maximum order of the polynomials: N. This will depend on the number of modes needed as will subsequently be discussed. A single set of Legendre polynomials cannot span both the water and bottom sediment, since it will not have the required derivative discontinuity at water-sediment interface. It is advisable to use the domain decomposition method of Li and Gottleib[18].

The set $\{P_k(\xi): k = 0, N; -1 \le \xi \le 1\}$ of Legendre polynomials is employed where the normalization is determined by $P_k(1) = 1$ resulting in the values $P_k(-1) = (-1)^k$ and the integrals $\int_{-1}^{1} P_k^2(\xi)d\xi = 2/(2k+1)$ for $k = 0, N$.

The domain decomposition takes the form

$$\psi_w(z) = \sum_{k=0}^{N} c_k P_k(\xi_w), \ 0 \le z \le h, \ \xi_w = (2/h)[z - (h/2)], \tag{8}$$

and

$$\psi_b(z) = \sum_{k=0}^{N} d_k P_k(\xi_b), \ h \le z \le H, \ \xi_b = [2/(H-h)]\{z - [(H+h)/2]\}. \tag{9}$$

The boundary conditions $\psi_w(0) = 0$ and $\psi_b(H) = 0$ and the interface conditions $\psi_w(h^-) = \psi_b(h^+)$ and $[1/\rho(h^-)]d\psi_w(h^-)/dz = 1/\rho(h^+)]d\psi_b(h^+)/dz$ yield four equations that determine four of the coefficients of the two highest order polynomials. The solutions of these equations yield expressions for the higher order coefficients in terms of the lower order coefficients and is a form of basis recombination[6].

The coefficients of the highest order polynomials are given by

$$c_N = \sum_{k=0}^{N-2}(e_{N,k}c_k + f_{n,k}d_k),$$

$$d_N = \sum_{k=0}^{N-2}(g_{N,k}c_k + h_{n,k}d_k),$$

$$c_{N-1} = \sum_{k=0}^{N-2}([e_{N,k} + (-1)^{k+N}]c_k + f_{n,k}d_k),$$

$$d_{N-1} = -\sum_{k=0}^{N-2}(g_{N,k}c_k + [h_{n,k} + 1]d_k)$$

where

$$e_{N,k} = [2(r_w + r_b)]^{-1}\{r_b(-1)^{N-1}Nk - r_w[N(N-1)(-1)^{k+N} + k(k+1)]/N^2\},$$

$$f_{N,k} = [2(r_w + r_b)]^{-1}\{r_b Nk - r_b[N(N-1)(-1)^N + k(k+1)(-1)^k]/N^2\},$$

$$g_{N,k} = [2(r_w + r_b)]^{-1}\{r_w Nk - r_w[N(N-1)(-1)^k + k(k+1)(-1)^N]/N^2\},$$

$$h_{N,k} = [2(r_w + r_b)]^{-1}\{r_w(-1)^{N-1}Nk - r_b[N(N-1) + k(k+1)(-1)^{k+N}]/N^2\}$$

with $r_w = 1/[\rho(h^-)h]$, $r_b = 1/[\rho(h^+)(H-h)]$ and $Nk = (-1)^N + (-1)^k$. These closed form expressions are found by solving the four equations determined by the boundary

and interface conditions. The solution process is simplified by assuming N is even or odd. The resulting solutions are consolidated, above, using factors of $(-1)^N$.

The boundary and interface conditions reduce the number of unknown expansion coefficients by four. Substitution of the expressions for c_N, c_{N-1}, d_N and d_{N-1} into Eq. (8) and Eq. (9) yields a test function ψ defined by

$$\psi_w(z) = \sum_{k=0}^{N-2} Lwc_k(\xi_w)c_k + \sum_{k=0}^{N-2} Lwd_k(\xi_w)d_k \; , \tag{10}$$

$$\psi_b(z) = \sum_{k=0}^{N-2} Lbc_k(\xi_b)c_k + \sum_{k=0}^{N-2} Lbd_k(\xi_b)d_k \; , \tag{11}$$

where

$$Lwc_k(\xi_w) = P_k(\xi_w) + [e_{N,k} + (-1)^{k+N}]P_{N-1}(\xi_w) + e_{N,k}P_N(\xi_w) \, ,$$
$$Lwd_k(\xi_w) = f_{N,k}[P_{N-1}(\xi_w) + P_N(\xi_w)]$$

and

$$Lbc_k(\xi_b) = g_{N,k}[P_N(\xi_b) - P_{N-1}(\xi_b)_w] \, ,$$
$$Lbd_k(\xi_b) = P_k(\xi_b) + h_{N,k}P_N(\xi_b) - [h_{N,k} + 1]P_{N-1}(\xi_b) \, .$$

The remaining expansion coefficients can be chosen as arbitrary complex numbers resulting in a wide range of complex test functions. The two highest order Legendre polynomials are not eliminated from Eq. (10) and Eq. (11); they are included with coefficients that assure that the boundary and interface conditions are satisfied. The actual choice of the expansion coefficients is determined in the application of the Galerkin method to the approximation of the solutions of Eq. (2).

The spanning set used in the Legendre-Galerkin method is obtained by singling out the real polynomials multiplying the complex expansion coefficients in Eq.'s (10) and (11): Define the following real test functions

$$\psi c_k(z) = \begin{cases} Lwc_{k-1}(\xi_w), \; 0 \le z \le h \\ Lbc_{k-1}(\xi_b), \; h < z \le H \end{cases} \, , \quad \psi d_k(z) = \begin{cases} Lwd_{k-1}(\xi_w), \; 0 \le z \le h \\ Lbd_{k-1}(\xi_b), \; h < z \le H \end{cases} \, ,$$

where $k = 1, N-1$ (note the index translation by 1). The Legendre-Galerkin spanning set is the union of the two sets $\{\psi c_k : k = 1, N-1\}$ and $\{\psi d_k : k = 1, N-1\}$ with a total of $M = 2(N-1)$ functions. The unknown expansion coefficient vectors $\mathbf{c} = col(c_{k-1} : k = 1, N-1)$ and $\mathbf{d} = col(d_{k-1} : k = 1, N-1)$ are stacked into a single $M = 2N - 2$ dimensional column vector $\mathbf{u} = col(\mathbf{c}, \mathbf{d})$. It will be convenient to sequentially re-index the Legendre-Galerkin spanning set: $\{\psi_j : j = 1, M\}$.

The size of the spanning set determines the number of eigenvalues and eigenvectors of Eq. (4). Consequently, $M = 2N - 2$ needs to be larger than the required number of modes, including the propagating and bottom interacting modes. A strategy that favors accuracy over the size of the calculation is to take M to be twice the required number of modes. Then retain half of the resulting approximations, of the horizontal wave numbers and normal modes, in the computation of the acoustic field. The maximum order of the Legendre polynomials $N = (M + 2)/2$ is proportional to the required number of modes.

The linear independence of the Legendre-Galerkin spanning set can be characterized as follows: Suppose a function is defined by $f(z) = \sum_{j=1}^{M} u_j \psi_j(z)$. If the expansion coefficients are uniquely determined by f, then the spanning set is linearly independent. (The only way to expand the zero function is with all zero coefficients.) The expansion can be integrated to obtain

$$\int_0^H \frac{f(z)\psi_m(z)}{\rho(z)} dz = \sum_{j=1}^{M} u_j \int_0^H \frac{\psi_j(z)\psi_m(z)}{\rho(z)} dz . \qquad (12)$$

Define the vector $\mathbf{f}$ by the entries $f_m = \int_0^H [f(z)\psi_m(z)/\rho(z)]dz$. Equation (12) becomes $\mathbf{Bu} = \mathbf{f}$, where the matrix $\mathbf{B}$ is the same as the one in Eq. (4). The expansion coefficients are uniquely determined if $\mathbf{B}$ is invertible.

Two assumptions have been made in the development of the Legendre-Galerkin method. They are the linear independence of the spanning set and the diagonalizability of the resulting matrix $\mathbf{C}$. These assumptions are both tested by failure or success in the numerical implementation.

When the Cholesky decomposition of the real symmetric matrix $\mathbf{B}$ fails, the matrix is not positive definite[11] (to within round off error). Otherwise, $\mathbf{B}$ is positive definite and the decomposition $\mathbf{B} = \mathbf{LL}^T$ can be used solve linear equation $\mathbf{Bu} = \mathbf{f}$ by two applications of backsubstitution. The matrix $\mathbf{B}$ is invertible and the Legendre-Galerkin spanning set is a basis.

The convergence of the Jacobi iteration, used to find the eigenvalues of the matrix $\mathbf{C} = \mathbf{L}^{-1}\mathbf{A}(\mathbf{L}^T)^{-1}$, is a numerical test of its diagonalizability. This requirement is necessary for the Legendre-Galerkin approximation to be consistent with the diagonalizability of the differential operator in Eq. (1).

The implementation of the Legendre-Galerkin method in FORTRAN allows for the numerical verification of the assumptions just described. When the Legendre-Galerkin method is implemented in a higher level language, like Matlab, this type of diagnostic information is not always available.

The matrices **A** and **B** in Eq. (4) are computed using Gauss-Legendre integration with $N+2$ quadrature points in each sound speed, attenuation and density layer in both the water layer and bottom sediment layer. The matrices **A** and **B** have dimension MxM, with four block sub-matrices of size $(N-1)x(N-1)$. The unknown expansion coefficients are found by solving the generalized matrix eigenvalue problem in Eq. (4). This provides the approximation of the eigenvalues and eigenfunctions of Eq. (2).

The errors in the approximate eigenvalues $\left|\lambda_m - \mu_m\right|$, $m = 1, M$ are expected[19] to be exponentially small for the first third, $m = 1, M/3$. Care must be taken since the Mth error is bound[19] to be very large. This large error is omitted from the presentation of Li and Gottleib[18].

A. Example:

An example of the acoustic field, approximated by the Legendre-Galerkin method, serves to numerically verify the algebraic construction of the Legendre polynomial basis. The example is taken from the previous report[2] that shows errors, in eigenvalues, computed with two Galerkin methods. The errors were found by comparison with a benchmark calculation.

The dimensions of the water and bottom layers are determined by $h = 200$ and $H = 1000$ meters. The circular frequency is $\omega = 2\pi f$ where $f = 250\ Hz$ (this is a correction[2]). The water and bottom layers have constant densities. The wave number squared $k^2(z)$ is assumed to be piecewise linear within both the water and bottom layers. The wave number $k(z)$ is determined by the values in Tab. 1.

z (m)	c (m/s)	α (dB/wave length)	ρ (gm/cm^3)
0	1520	.00	1.0
20	1520	.00	1.0
30	1480	.00	1.0
200	1480	.00	1.0
200	1600	.05	1.5
1000	2000	.20	1.5

Table 1. Acoustic parameters of the example waveguide. Differences between different depths indicate gradients. The double entry at 200 m represents the discontinuity in the parameters at the water-bottom interface. The attenuation gradient in the bottom layer is a modest artificial absorber.

The acoustic field is calculated using Eq. (7) for a source at $z_s = 50\ m$. The receiver is at a depth of $z = 100\ m$, on the range interval $r = 10$ to $4{,}000\ m$. There are around 50 propagating modes and 50 bottom interacting modes, so a values of $M = 198$ (N=100) and $MS = 100$ are used in the Legendre-Galerkin calculation. The problem is also solved using the Fourier-Galerkin method[2] with $M = 198$. Transmission loss[1] is computed using

$TL(r, z) = -20 \log_{10}(4\pi |p(r, z)|)$, with both methods. The comparison of two methods, in terms of transmission loss, is shown in Fig. 2.
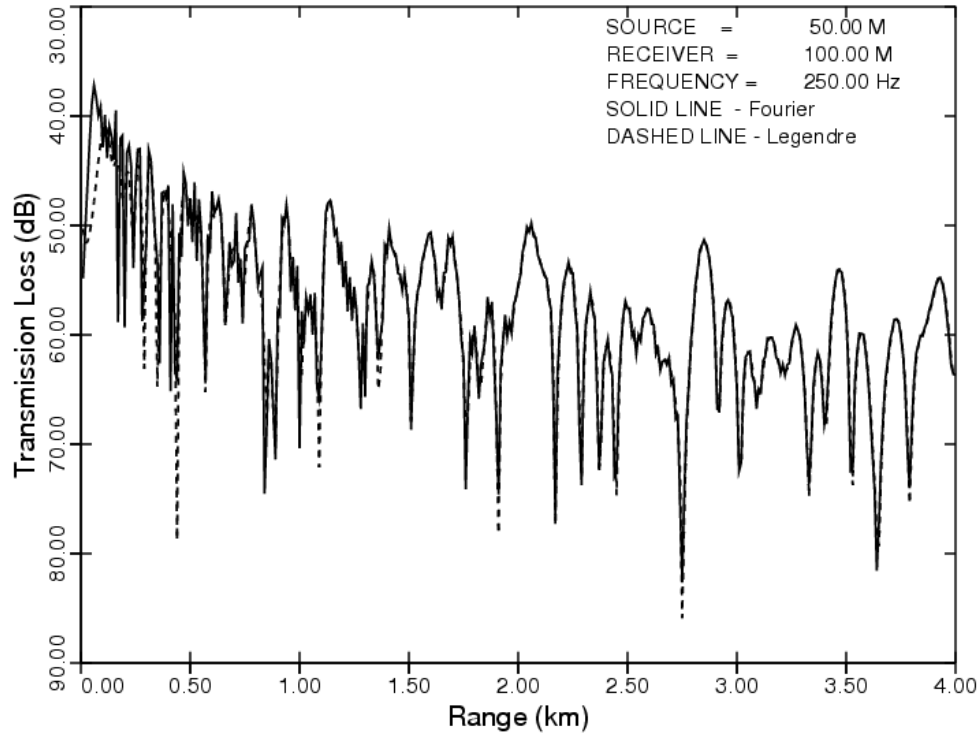

Figure 2. Transmission loss versus range on the interval .01 to 4.0 km is computed using the Fourier-Galerkin and the Legerdre-Galerkin methods. The agreement of these two methods is a test of the validity of the construction of the Legendre-Galerkin basis.

V. CONCLUSION:

The Legendre-Galerkin method provides a relatively simple procedure for solving the non-selfadjoint differential eigenvalue problem in Eq. (1). The method uses a basis that can be constructed algebraically. The Fourier-Galerkin method requires the solution of a transcendental equation, in construction of the orthogonal basis.

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References:

[1]F. B. Jensen, W. A. Kuperman, M. B. Porter and H. Schmidt, *Computational Ocean Acoustics*, pp. 338-340 (Springer-Verlag, New York, 2011).

[2]R. B. Evans, "The rate of convergence and error distribution of Galerkin approximations to eigenvalues in underwater acoustics," Proc. of Meetings on Acoust. (POMA), **19**, 070081 (2013) On the web at: http://dx.doi.org/10.1121/1.4801391.

[3]D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, pp. 29-32 (Society for Industrial and Applied Mathematics, Philadelphia, 1977).

[4]C. Canuto, M. Y. Hussaini, A. Quartreoni and T. A. Zang, *Spectral Methods – Fumdamentals in Single Domains*, pp. 3-24 (Springer-Verlag, New York, 2006).

[5]R. B. Evans, "The continuation of the normal mode solution of the underwater acoustic wave equation," Wave Motion, **20**, 13-20 (1994).

[6]J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Second Edition (Revised), Chap. 3 especially pp. 63-70 (Dover, New York, 2001).

[7]B. Friedman, *Principles and Techniques of Applied Mathematics*, p. 11 (Dover, New York, 1990).

[8]G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, pp. 24-27 (Wellesley-Cambridge Press, Wellesley, MA, 1997).

[9]F. Chatelin, *Spectral Approximations of Linear Operators*, p. 170 (Academic Press, New York, 1983).

[10]I. Babuška and J. Osborn, "Eigenvalue Problems," in *Handbook of Numerical Analysis*, Vol. II, Finite Element Methods (Part 1), edited by G. Ciarlet and J. L. Lions, pp. 663-667, 692-700 (Elsevier, Amsterdam, The Netherlands, 1991).

[11]W. H. Press, S. A. Teukolsky, W. T Vettering and B. P. Flannery, *Numerical Recipes in FORTRAN*, Second Edition, pp. 89-91, 455 (Cambridge University Press, New York, 1992).

[12]R. A. Horn and C. R. Johnson, *Matrix Analysis*, pp. 211-212 (Cambridge University Press, New York, 1985).

[13]D. S. Watkins, *Fundamentals of Matrix Computations*, pp. 353-354 (John Wiley and Sons, New York, 1991).

[14]P. J. Anderson and G. Loizou, "A Jacobi type method for complex symmetric matrices," Numer. Math. 25, 347-363 (1976).

[15]A. T. I. Admou and R.V. Craster, "Spectral methods for modeling guided waves in elastic media, " J. Acoust. Soc. Am. **116** (3), 1524-1535 (2004).

[16]F. Hernado Quintanilla, M. J. S. Lowe and R.V. Craster, "Modeling guided elastic waves in generally anisotropic media using a spectral collocation method," J. Acoust. Soc. Am. **137** (3), 1180-1194 (2015).

[17]E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, p. 310 (McGraw-Hill, New York, 1955).

[18]M. S. Min and D. Gottlieb, "Domain decomposition spectral approximations for an eigenvalue problem with a piecewise constant coefficient," SIAM J. Numer. Anal. **43**, 502-520 (2005).

[19]C. Bernardi and Y. Maday, "Spectral Methods," in *Handbook of Numerical Analysis*, Vol. V, Techniques of Scientific Computing (Part 2), edited by G. Ciarlet and J. L. Lions, 209-485 (Elsevier, Amsterdam, The Netherlands, 1997) pp. 281-285.