Implementation of a new root finder for Kraken

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1 INTRODUCTION

In underwater acoustics or more precisely in shallow water propagation, the determination of the modes propagating in the water duct is critical. Several approaches in solving the propagation problem in an acousto-elastic environment have been discussed. The first approach which is the most widely used is the KRAKEN normal mode program [1] which divides the environment into layers. Recently another model called ORCA [2] based on Airy function solutions was proposed. While KRAKEN uses the secant method for finding modes, ORCA uses a special mode finding algorithm which improves its robustness. But difficulties still remain because of the presence of branch line integrals. For example, ORCA requires the use of false bottoms which complicate the mode solution.

A new root finder for KRAKEN which makes the code more robust is presented in this paper. It is based on winding number integrals and does not require the use of false bottoms. Consequently robust and accurate solutions of the depth-dependent wave equation can be obtained. In addition to this new root finder, a simple modification of the characteristic equation is used to find the modes which are close to a branch points. Some examples of the behaviour as a function of frequency of a mode which is closed to a branch point will be presented.

2 WINDING NUMBER INTEGRALS

This section closely follows the article published by Davies [3] and is just a summary of the basic principles.

Consider $f(z)$, an analytic function over a region in the complex plane except for n_q poles, φ an analytic function and C a closed contour on which f has no zeros or poles, then the winding number integral is defined by :

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{f'(z)}{f(z)} \varphi(z) dz = \sum_{k=1}^{p} n_k \varphi(a_k) - \sum_{l=1}^{q} n_l \varphi(b_l)$$

where a_k et b_l are the zeros and poles of f inside the contour C with the multiplicity orders n_k and n_l. This result also called the argument principle is a consequence of the residue theorem.

Let suppose that f has no poles inside C, that all the zeros are simple and take as function φ the following function : $\varphi(z) = z^n$. Under this hypothesis, the winding number integral will lead to the following relation :

$$S_n = \frac{1}{2\pi \mathrm{i}} \int_C z^n \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{N} a_k^n$$

This means that the latter equation provides the knowledge of the number of zeros, the sum of the roots, the sum of the squares of the roots and etc... It is then possible to use Newton's relation [4] to determine the coefficients of the polynomial which have these roots. The problem of finding the zeros of function f is then reduced to the canonical problem of finding the roots of a polynomial.

3 Branch cuts

The propagation problem is an improper Sturm-Liouville problem. Consequently there is a discrete spectrum corresponding to the modes and a continuous spectrum which is related to the branch cuts. Branch cuts arise from functions of the following type : $K(\xi) = \sqrt{k^2 - \xi^2}$. Their presence complicates the design of the contours used with the winding number integral method since they must not cross them. The Pekeris cut cut can be used :

$$K(\xi) = \sqrt{k^2 - \xi^2} = -\mathrm{i}\sqrt{-\mathrm{i}(k-\xi)}\sqrt{-\mathrm{i}(k+\xi)} \quad \text{for } e^{-i\omega t} \text{ dependence}$$
$$K(\xi) = \sqrt{k^2 - \xi^2} = -\mathrm{i}\sqrt{+\mathrm{i}(k-\xi)}\sqrt{+\mathrm{i}(k+\xi)} \quad \text{for } e^{+i\omega t} \text{ dependence}$$

but the modified version of the EJP cuts [4] is prefered because it involves "physical" Riemann sheets.

4 Implementation of the new root finder

A more complete version of the structure of the new root finder can be found in reference [7]. The procedure used to construct the root finder is the following. Winding number integrals are only used for the first mesh in order to save cputime. It can be seen as a tool that provides good initial guesses to the secant method. The complex plane is divided into areas where the roots are to be found. This partitionning is realized in order to control the maximum number of modes which will be evaluated inside one contour. It is performed by means of the effective depth concept which is used to get an approximation of the number of modes that can be found. This concept was extended to determine the number of modes whose sound speed is less than the sound speed in water when the bottom is made of elastic layers. An illustration of the results provided by the new root finder follows :

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KRAKENZ-Love problem

Frequency =  100.00     NMEDIA =   2


    C-LINEAR approximation to SSP
    Attenuation units: dB/mkHz
    VACUUM


     Z          ALPHAR     BETAR      RHO       ALPHAI     BETAI


         ( Number of pts =  1000  RMS roughness =  0.000E+00 )
     0.00      1500.00      0.00     1.00       0.0000    0.0000
   100.00      1500.00      0.00     1.00       0.0000    0.0000

         ( Number of pts =   250  RMS roughness =  0.000E+00 )
   100.00      1650.00    400.00     1.90       0.2000    0.4000
   125.00      1650.00    400.00     1.90       0.2000    0.4000

                                 ( RMS roughness =  0.000E+00 )
    ACOUSTO-ELASTIC half-space
   125.00      1800.00    600.00     2.00       0.5000    1.0000

CLOW =   350.00      CHIGH =   4000.0
RMIN, RMAX =   0.500000000000000        1000.00000000000

Number of sources   =            1
  70.00000

Number of receivers =            1
  70.00000

Mesh multiplier   CPU seconds
       1            1.38
       2           0.900E-01
       4           0.137
```

I	K	ALPHA	PHASE SPEED
1	1.769048619	-0.5254541008E-02	355.1731276
2	1.564918878	-0.4629575835E-02	401.5023012
3	1.547237345	-0.4703582270E-02	406.0905929
4	1.517575006	-0.4830383694E-02	414.0279908
5	1.475566473	-0.5017624596E-02	425.8151308
6	1.420621531	-0.5280505306E-02	442.2842517
7	1.352007838	-0.5648451693E-02	464.7299470
8	1.269430079	-0.6189979409E-02	494.9611175
9	1.176691042	-0.7158749501E-02	533.9706928
10	1.098289062	-0.8541673473E-02	572.0884895
11	1.023047247	-0.1695520089E-01	614.1637471
12	0.8843540613	-0.2972435984E-01	710.4830047
13	0.7101347038	-0.5852239795E-01	884.7878119
14	0.5186426805	-0.1345577218	1211.467074
15	0.4178677301	-0.1000692274E-04	1503.630181
16	0.4148148900	-0.3621321931E-04	1514.696183
17	0.4096651752	-0.6993185409E-04	1533.736741
18	0.4023339943	-0.1044807112E-03	1561.683923
19	0.3927347813	-0.1465731845E-03	1599.854560
20	0.3809444112	-0.2769357161E-03	1649.370649
21	0.3696161565	-0.1521413798E-02	1699.921715
22	0.3627194018	-0.1101521405E-02	1732.244064
23	0.3474937937	-0.1409253781E-02	1808.143173
24	0.3343206274	-0.1136562491E-01	1879.389063
25	0.3214665499	-0.5127952232E-02	1954.537823
26	0.2935955734	-0.4515894038E-02	2140.081757
27	0.2592216614	-0.6420814404E-02	2423.865843
28	0.2191096548	-0.5953527667E-01	2867.598561
29	0.2123856585	-0.1085559467E-01	2958.384926

With the same environment file, a Krakenc run was made. Only 17 modes were found by Krakenc which omits the first three propagative modes.

5 Root close to a branch point

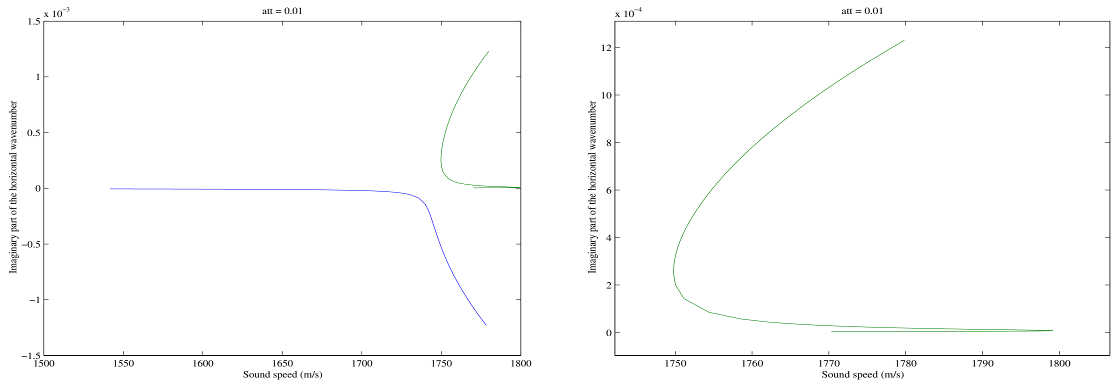
When refining the mesh, a mode can become evanescent i.e. going to the other Riemann sheet. This discontinuity introduced by the branch cut is critical for the secant method which will fail in this case. Therefore there is a need for a method which will enable the finding of a root whose sound speed is close and superior to the sound speed associated with the branch point. This is easily done by changing the characteristic equation in the following way.

Instead of solving : $f(\xi, K_1(\xi), K_2(\xi), ..., K_j(\xi), ..., K_n(\xi)) = 0$

One solve : $F(\xi) = 0$ with

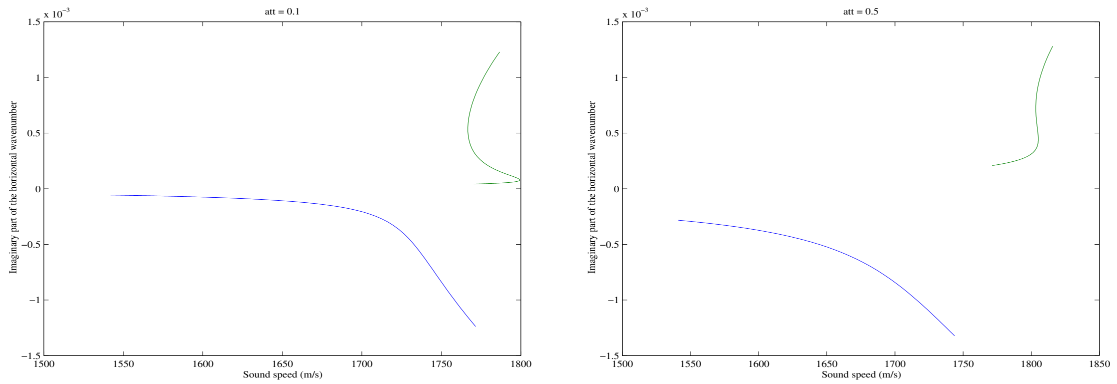
$$F(\xi) = f(\xi, K_1(\xi), K_2(\xi), ..., K_j(\xi), ..., K_n(\xi)) \times f(\xi, K_1(\xi), K_2(\xi), ..., -K_j(\xi), ..., K_n(\xi))$$

The function $F(\xi)$ is analytic in the area close to the branch point $K_j(\xi)$ and there is no discontinuity. By means of this technique it is then possible to look at the dispersion curves of modes in the area which is close to a branch point. Some examples are given in the following figures. The bottom is a semi-infinite fluid medium with a sound speed of 1800 m/s. The depth of the water column is 100 m with a sound speed of 1500 m/s. Different dispersion curves of mode 1 are presented with different values of the attenuation in the fluid medium. The frequency increases from 6.1 Hz to 7.7 Hz.



(a) Dispersion curves for mode 1 (b) Zoom for mode 1

Figure 1: Dispersion curves for Mode 1 (attenuation fixed to 0.01 dB/m/kHz)



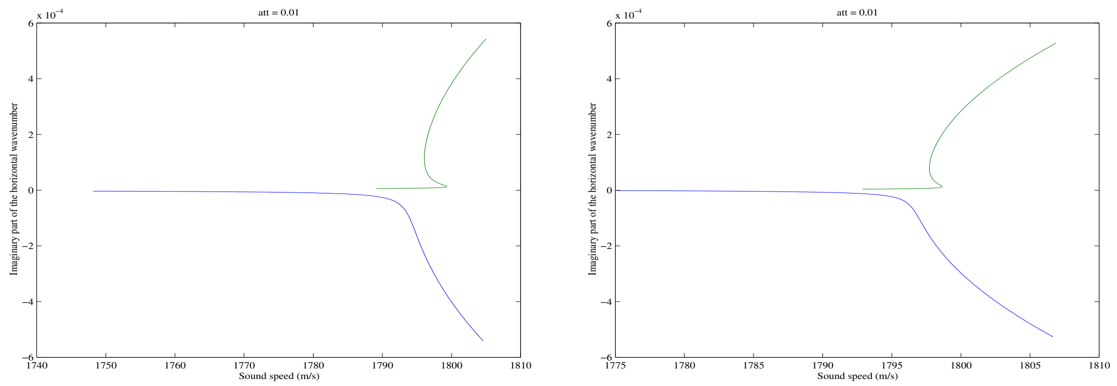
(a) Attenuation 0.1 dB/m/kHz (b) Attenuation 0.5 dB/m/kHz

Figure 2: Dispersion curves for Mode 1

Surprisingly the real part of the mode is not increasing with frequency. It is then possible to have an evanescent mode whose real part corresponds to a propagating mode.

Figures 2(a) et 2(b) show the influence of attenuation on the dispersion curves of mode 1. Attenuation acts as a smoothing process.

Next figures show the behaviour of modes 2 and 3. For mode 2, the frequency increases from 20 Hz to 21 Hz. For mode 3, the frequency increases from 33.5 Hz to 34.5 Hz.



(a) Mode 2

(b) Mode 3

Figure 3: Dispersion curves for modes 2 and 3

Modes 2 and 3 exhibits the same behaviour as mode 1.

References

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